Chapter 1

Fundamental Concepts

Exercise 1.1: Consider a sample space S and assume for a given subset B that $P(B) > 0$. Show that the conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{1.1}$$

satisfies the axioms of probability.

Exercise 1.2: Show that

$$P(A \cup B) \;=\; P(A) + P(B) - P(A \cap B)\,.$$

(Express $A \cup B$ as the union of three disjoint sets.)

Exercise 1.3: A beam of particles consists of a fraction 10^{-4} electrons and the rest photons. The particles pass through a double-layered detector which gives signals in either zero, one or both layers. The probabilities of these outcomes for electrons (e) and photons (γ) are

$$\begin{array}{lcl} P(0\,|\,\mathrm{e}) = 0.001 & \text{and} & P(0\,|\,\gamma) = 0.99899 \\ P(1\,|\,\mathrm{e}) = 0.01 & & P(1\,|\,\gamma) = 0.001 \\ P(2\,|\,\mathrm{e}) = 0.989 & & P(2\,|\,\gamma) = 10^{-5}\,. \end{array}$$

(a) What is the probability for a particle detected in one layer only to be a photon?

(b) What is the probability for a particle detected in both layers to be an electron?

Exercise 1.4: Suppose a random variable x has the p.d.f. $f(x)$. Show that the p.d.f. for $y = x^2$ is

$$g(y) \;=\; \frac{1}{2\sqrt{y}} f(\sqrt{y}) \;+\; \frac{1}{2\sqrt{y}} f(-\sqrt{y})\,. \tag{1.2}$$

Exercise 1.5: Suppose two independent random variables x and y are both uniformly distributed between zero and one, i.e. the p.d.f. $g(x)$ is given by

$$g(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise}\,, \end{cases} \tag{1.3}$$

and similarly for the p.d.f. $h(y)$.

(a) Using SDA equation (1.35), show that the p.d.f. $f(z)$ for $z = xy$ is

$$f(z) = \begin{cases} -\log z & 0 < z < 1 \\ 0 & \text{otherwise}. \end{cases} \tag{1.4}$$

(b) Find the same result using SDA equations (1.37) and (1.38) by defining an additional function, $u = x$. First, find the joint p.d.f. of z and u. Integrate this over u to find the p.d.f. for z.

(c) Show that the cumulative distribution of z is

$$F(z) = z(1 - \log z)\,. \tag{1.5}$$

Exercise 1.6: Consider a random variable x and constants α and β. Show that

$$\begin{aligned} E[\alpha x + \beta] &= \alpha E[x] + \beta\,, \\ V[\alpha x + \beta] &= \alpha^2 V[x]\,. \end{aligned} \tag{1.6}$$

Exercise 1.7: Consider two random variables x and y.

(a) Show that the variance of $\alpha x + y$ is given by

$$\begin{aligned} V[\alpha x + y] &= \alpha^2 V[x] + V[y] + 2\alpha \mathrm{cov}[x,y] \\ &= \alpha^2 V[x] + V[y] + 2\alpha\rho\sigma_x\sigma_y\,, \end{aligned} \tag{1.7}$$

where α is any constant value, $\sigma_x^2 = V[x]$, $\sigma_y^2 = V[y]$, and the correlation coefficient is $\rho = \mathrm{cov}[x,y]/\sigma_x\sigma_y$.

(b) Using the result of (a), show that the correlation coefficient always lies in the range $-1 \le \rho \le 1$. (Use the fact that the variance $V[\alpha x + y]$ is always greater than or equal to zero and consider the cases $\alpha = \pm\sigma_y/\sigma_x$.)

Exercise 1.8: Suppose $\mathbf{x} = (x_1, \ldots, x_n)$ is described by the joint p.d.f. $f(\mathbf{x})$, and the variables $\mathbf{y} = (y_1, \ldots, y_n)$ are defined by means of a linear transformation,

$$y_i = \sum_{j=1}^{n} A_{ij} x_j. \tag{1.8}$$

Assume that the inverse transformation $\mathbf{x} = A^{-1}\mathbf{y}$ exists.

(a) Show that the joint p.d.f. for $\mathbf{y}$ is given by

$$g(\mathbf{y}) = f(A^{-1}\mathbf{y})\,|\det(A^{-1})|. \tag{1.9}$$

(b) Find $g(\mathbf{y})$ for the case where A is orthogonal, i.e. $A^{-1} = A^T$.