Lecture 3 outline

1. **The Monte Carlo method**
2. **Random number generators**
3. **The transformation method**
4. **The acceptance-rejection method**
5. **Accuracy of MC**
6. **Uses in particle physics**

The Monte Carlo method

What it is: a numerical technique for calculating probabilities and related quantities using sequences of random numbers.

The usual steps:

(1) Generate sequence $r_1, r_2, \ldots, r_m$ uniform in $[0, 1]$.

(2) Use this to produce another sequence $x_1, x_2, \ldots, x_n$ distributed according to some pdf $f(x)$ in which we're interested. (N.B. x can be a vector.)

(3) Use the x values to estimate some property of $f(x)$, e.g. fraction of x values with $a \le x \le b$ gives $\int_a^b f(x)\,dx$.

$\Rightarrow$ MC calculation = integration (at least formally)

Usually trivial for 1-d: $\int_a^b f(x)\,dx$ obtainable by other methods.

MC more powerful for multidimensional integrals.

MC x values = 'simulated data'

$\rightarrow$ use for testing e.g. statistical procedures.

Random number generators

Goal: uniformly distributed values in $[0, 1]$.

Toss coin for e.g. 32 bit number ... (too tiring).

$\Rightarrow$ 'random number generator'

= computer algorithm to generate $r_1, r_2, \ldots, r_n$.

Example: the multiplicative linear congruential generator (MLCG)

$$n_{i+1} = (a n_i) \bmod m \,, \qquad \text{where}$$

n_i = integer

a = multiplier

m = modulus

n_0 = seed

N.B. mod = modulus (remainder), e.g. $27 \bmod 5 = 2$.

The n_i follow periodic sequence in $[1, m-1]$.

Example (cf. Brandt): $a = 3$, $m = 7$, $n_0 = 1$:

$$\begin{aligned}
n_1 &= (3 \cdot 1) \bmod 7 = 3 \\
n_2 &= (3 \cdot 3) \bmod 7 = 2 \\
n_3 &= (3 \cdot 2) \bmod 7 = 6 \\
n_4 &= (3 \cdot 6) \bmod 7 = 4 \\
n_5 &= (3 \cdot 4) \bmod 7 = 5 \\
n_6 &= (3 \cdot 5) \bmod 7 = 1 \leftarrow \text{sequence repeats!}
\end{aligned}$$

Choose a, m, to obtain long period (maximum = $m - 1$).

Random number generators (continued)

$r_i = \frac{n_i}{m}$ are in $[0,1]$ (0 and 1 excluded), but are they 'random'???

Choose a, m, so that the r_i pass various tests of randomness:

Uniform distribution in $[0,1]$

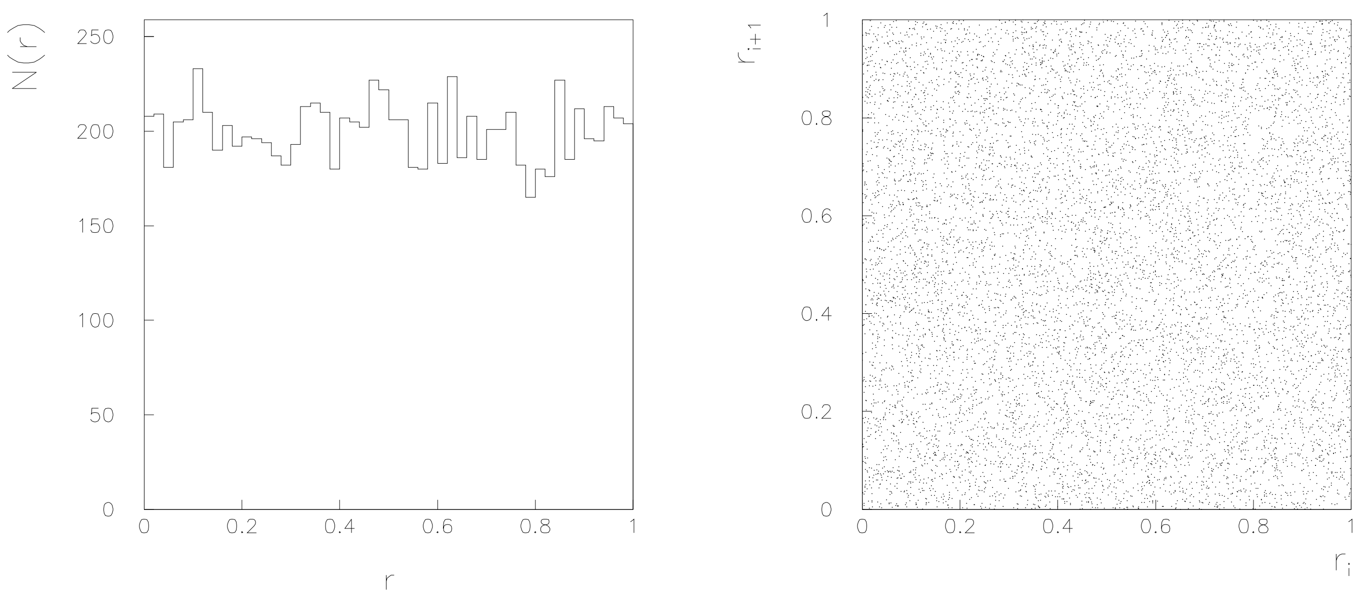
All pairs independent (no correlations)

e.g. L'Ecuyer, Commun. ACM 31 (1988) 742 suggests

$$a = 40692$$

$$m = 2147483399$$

Test with 10000 generated values:



Far better algorithms available e.g. **RANMAR**, period $\approx 2 \times 10^{43}$.

For more info see e.g.

F. James, Comput. Phys. Commun. 60 (1990) 111;

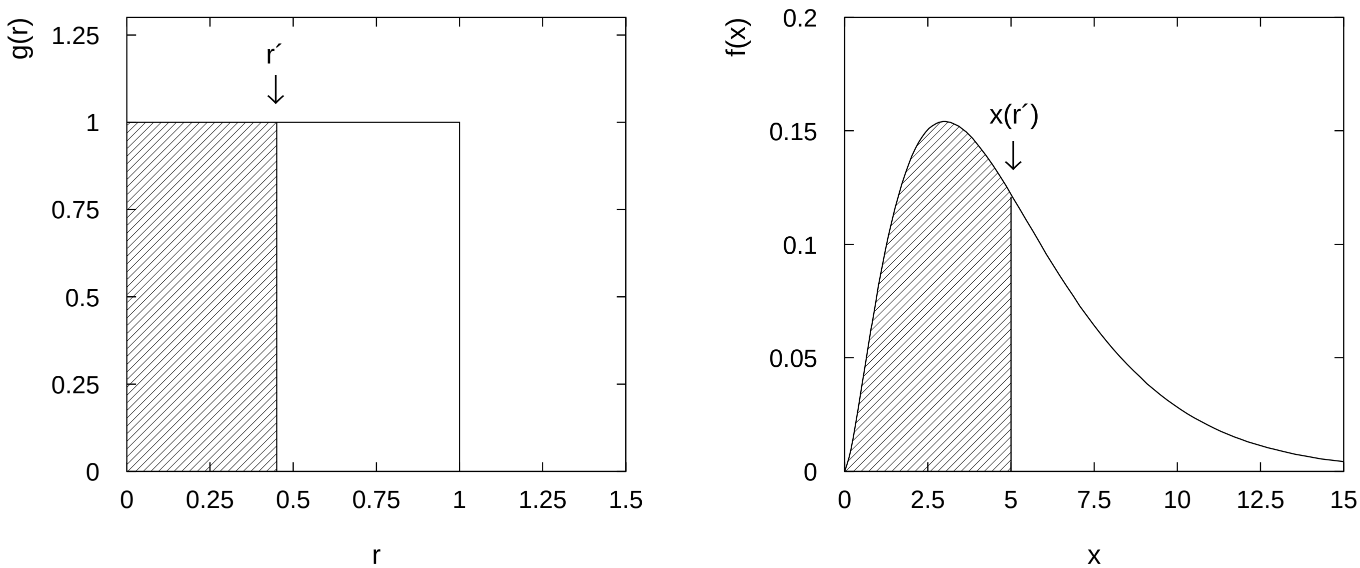
Brandt, chapter 4.

Program for generating uniform random numbers with **RANMAR**

```
      program TEST_RANMAR

      implicit        NONE

c   Needed for HBOOK routines

      integer         hsize
      parameter       (hsize = 100000)

      integer         hmemor (hsize)
      common  /pawc/  hmemor

c   Local variables

      character*80    outfile
      integer         i, icycle, istat
      integer         NTOTIN, NTO2IN, IJKLIN
      real            rvec(1)   ! vector of random nos. (here only 1)

c   Initialize HBOOK, open histogram file, book histograms, set seed.

      call HLIMIT (hsize)
      outfile = 'test_ranmar.his'
      call HROPEN (20, 'histog', outfile, 'N', 1024, istat)
      call HBOOK1 (1, 'uniform dist', 100, 0., 1., 0.)

      write (*, *) 'Enter initial seed between 0 and 900 000 000'
      read (*, *) IJKLIN
      NTOTIN = 0
      NTO2IN = 0
      call RMARIN(IJKLIN,NTOTIN,NTO2IN)       ! sets initial seed

c   Generate 10000 values, enter into histogram, then store histogram.

      do i = 1, 10000
        call RANMAR (rvec, 1)
        call HF1 (1, rvec(1), 1.)
      end do

      call HROUT (0, icycle, ' ')
      call HREND ('histog')

      stop
      END
```

The transformation method

Given $r_1, r_2, \ldots, r_n$ uniform in $[0,1]$, find $x_1, x_2, \ldots, x_n$ which follow $f(x)$ by finding a suitable transformation $x(r)$.



Require: $P(r \leq r') = P(x \leq x(r'))$

i.e. $\int_{-\infty}^{r'} g(r)\,dr = r' = \int_{-\infty}^{x(r')} f(x')\,dx' = F(x(r'))$

That is,

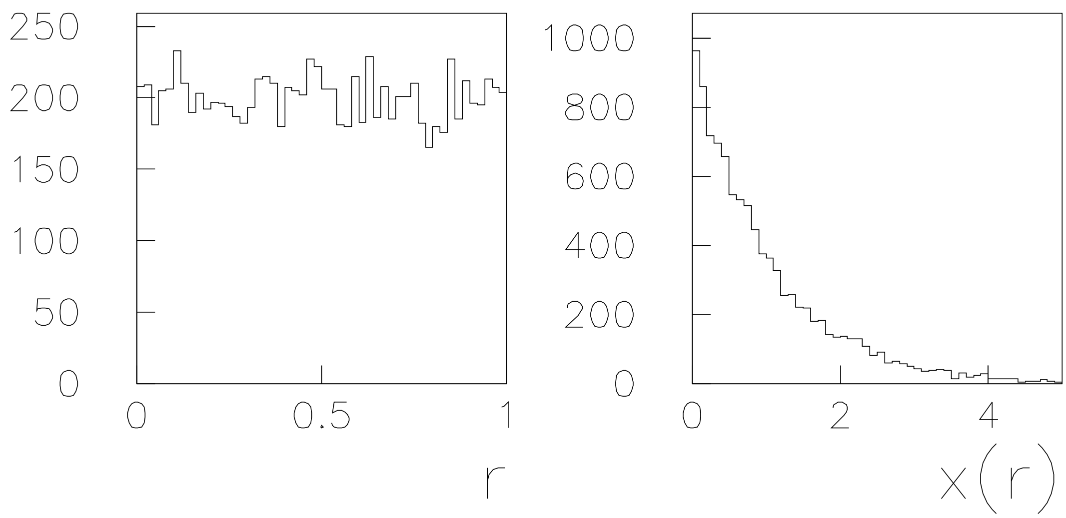
$$\boxed{\text{set } F(x(r)) = r \text{ and solve for } x(r).}$$

Example of the transformation method

Exponential pdf: $f(x;\xi) = \frac{1}{\xi} e^{-x/\xi} \qquad (x \geq 0)$

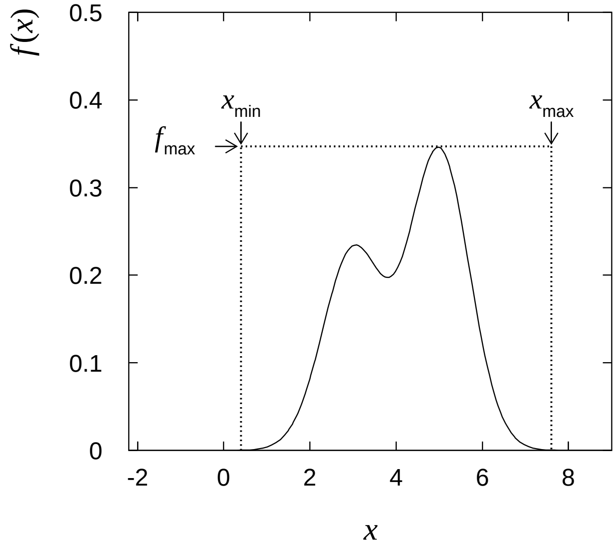
Set $\int_0^x \frac{1}{\xi} e^{-x'/\xi}\, dx' = r$ and solve for $x(r)$.

$\Rightarrow x(r) = -\xi \log(1-r) \qquad (x(r) = -\xi \log r$ works too.)



The acceptance-rejection method (von Neumann)

Enclose the pdf in a box:



(1) Generate a random number x, uniform in $[x_{\mathrm{min}}, x_{\mathrm{max}}]$, i.e. $x = x_{\mathrm{min}} + r_1(x_{\mathrm{max}} - x_{\mathrm{min}})$ where r_1 is uniform in $[0, 1]$.

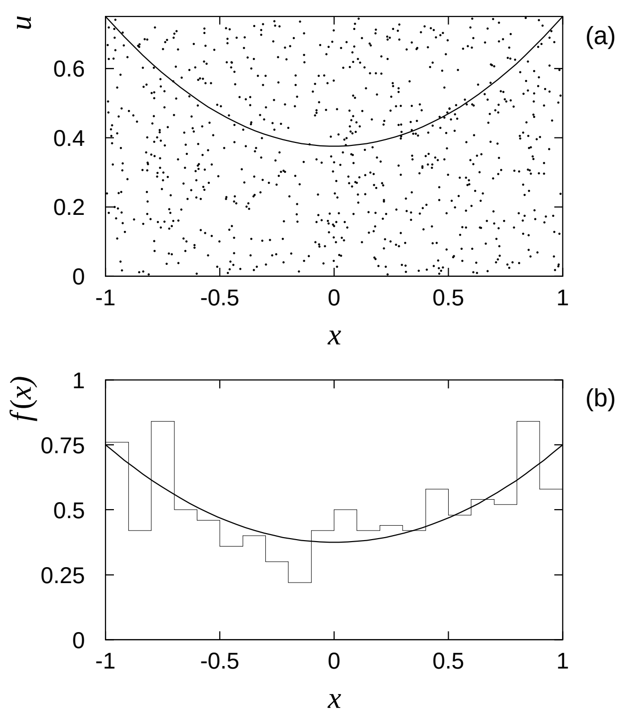
(2) Generate a second independent random number u uniformly distributed between 0 and f_{max}, i.e. $u = r_2 f_{\mathrm{max}}$.

(3) If $u < f(x)$, then accept x. If not, reject x and repeat.

Example:

$$f(x) = \frac{3}{8}(1 + x^2)$$

$$(-1 \leq x \leq 1)$$

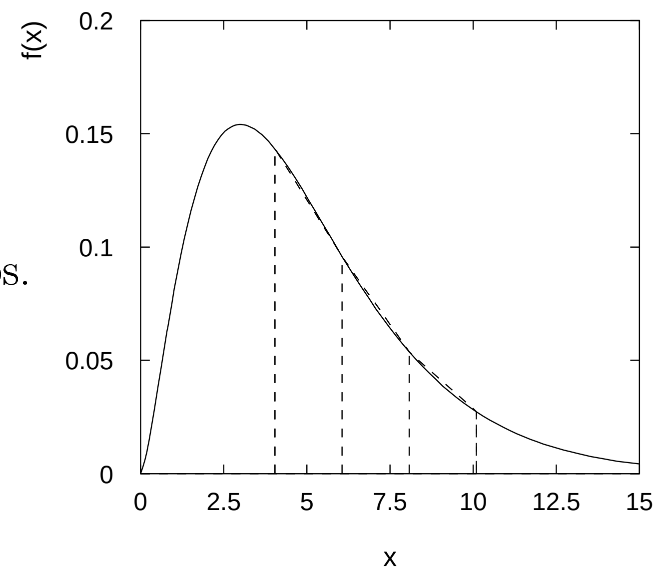


Accuracy of Monte Carlo

MC calculation = integration.

Compare to trapezoidal rule,

n = number of computing steps.



For 1-dimensional integral:

MC: $n \propto$ number of random values generated

accuracy $\propto 1/\sqrt{n}$

Trapezoid: $n \propto$ number of subdivisions

accuracy $\propto 1/n^2$

Trapezoid wins! But in d dimensions this becomes

MC: accuracy $\propto 1/\sqrt{n}$ $\leftarrow$ independent of d !

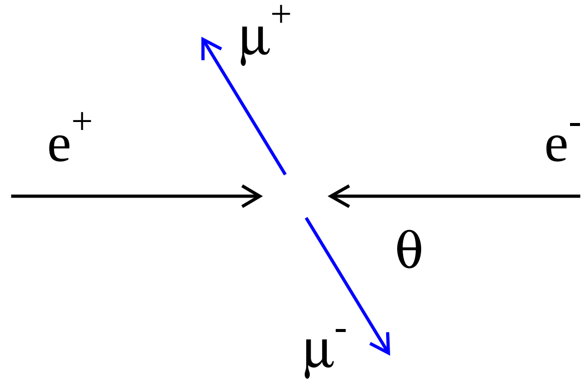
Trapezoid: accuracy $\propto 1/n^{2/d}$

MC wins for $d > 4$. Gaussian quadrature better than trapezoid, but for high enough d, MC always wins.

(See F. James, *Rep. Prog. Phys.* 43 (1980) 1145.)

Simple example:

$e^+e^- \to \mu^+\mu^-$



Generate θ and ϕ:

$$f(\cos\theta; A_{\mathrm{FB}}) \propto (1 + \tfrac{8}{3}A_{\mathrm{FB}}\cos\theta + \cos^2\theta)$$

$$g(\phi) = \frac{1}{2\pi}$$

Less simple examples:

$e^+e^- \to$ hadrons:	JETSET (PYTHIA)
	HERWIG
	ARIADNE
$pp \to$ hadrons:	ISAJET
	PYTHIA
	HERWIG
$e^+e^- \to WW$:	KORALW
	EXCALIBUR
	ERATO

Output = ‘events’, i.e. for each event, a list of final state particles and their momentum vectors.

Monte Carlo detector simulation

Takes as input the particle list and momenta from generator.

Simulate detector response:

- multiple Coulomb scattering (generate scattering angle)
- particle decays (generate lifetime)
- ionization energy loss (generate Δ)
- EM/hadronic showers
- production of signals, electronics response
- $\vdots$

Output = simulated raw data

$\rightarrow$ input to reconstruction software (track finding/fitting, etc.)

Uses:

Predict what you should see at 'detector level' given a certain hypothesis for 'generator level'. Compare with the real data.

Estimate various 'efficiencies' $= \dfrac{\#\text{ events found}}{\#\text{ events generated}}$

Programming package: GEANT

Lecture 3 summary

1. **The Monte Carlo method**

   numerical technique for computing probabilities (and things that can be related to probabilities) using random numbers,

   MC $\leftrightarrow$ integration,

   does not depend on interpretation of probability.

2. **Random number generators**

   produce sequence $r_1, r_2, \ldots, r_n$ uniform in $[0, 1]$,

   actually pseudorandom (i.e. reproducible if use same seed),

   simple algorithm: MLCG,

   better ones exist, e.g. `RANMAR`.

3. **The transformation method**

   set cumulative distribution $F(x) = r$, solve for $x(r)$,

   produces one value of x for each value of r.

4. **The acceptance-rejection method**

   must be able to enclose pdf $f(x)$ in a box,

   algorithm slow if pdf is sharply peaked.

5. **Accuracy of MC**

   accuracy $\propto 1/\sqrt{n}$

6. **Uses in particle physics**

   event generators

   detector simulation