G. COWAN

Statistical Data Analysis Discussion Notes Lecture Week 10

Problem Sheet 7 Sol'ns

1) $$f(x;\theta) = \frac{x^2}{2\theta^3} e^{-x/\theta} , \quad x \geq 0, \quad \theta > 0$$

$$E[x] = 3\theta , \quad V[x] = 3\theta^2$$

Given i.i.d. sample $x_1, \ldots, x_n$

For (a) – (c) assume n constant.

1a) $$L(\theta) = \prod_{i=1}^{n} \frac{x_i^2}{2\theta^3} e^{-x_i/\theta}$$

$$\ln L(\theta) = \sum_{i=1}^{n} \left[\ln x_i^2 - \ln 2 - \ln \theta^3 - \frac{x_i}{\theta} \right]$$

$$= -3n \ln \theta - \frac{1}{\theta} \sum_{i=1}^{n} x_i + C$$

$$\frac{\partial \ln L}{\partial \theta} = -\frac{3n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} x_i \overset{\text{set}}{=} 0$$

$$\Rightarrow \quad \hat{\theta} = \frac{1}{3n} \sum_{i=1}^{n} x_i$$

b) $E[\hat{\theta}] = E\left[\frac{1}{3n}\sum_{i=1}^{n} x_i\right]$

$$= \frac{1}{3n}\sum_{i=1}^{n} \underbrace{E[x_i]}_{\substack{\shortparallel \\ 3\theta}} = \theta$$

$$\Rightarrow \quad b = E[\hat{\theta}] - \theta = 0$$

$$V[\hat{\theta}] = V\left[\frac{1}{3n}\sum_{i=1}^{n} x_i\right]$$

$$= \frac{1}{9n^2}\sum_{i=1}^{n} \underbrace{V[x_i]}_{\substack{\shortparallel \\ 3\theta^2}} = \frac{\theta^2}{3n}$$

$$MVB = -\frac{\left(1+\frac{\partial b}{\partial \theta}\right)^2}{E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right]}$$

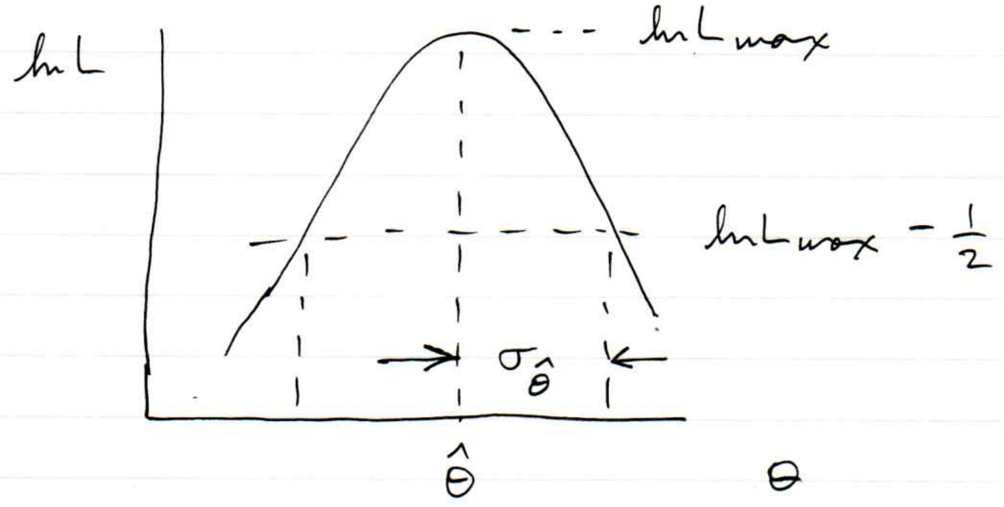
$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{3n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i$$

$$E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = \frac{3n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} \underbrace{E[x_i]}_{\substack{\shortparallel \\ 3\theta}} = -\frac{3n}{\theta^2}$$

$$\Rightarrow \quad MVB = -\frac{(1+0)^2}{\left(-\frac{3n}{\theta^2}\right)} = \frac{\theta^2}{3n}$$

$= $ same as $V[\hat{\theta}] \Rightarrow \hat{\theta}$ is <u>efficient</u>.

1c)



For (d)-(f), treat $n \sim \text{Poisson}(\nu)$ i.e. $P(n|\nu) = \frac{\nu^n e^{-\nu}}{n!}$

$$\nu = \alpha \theta^3 \qquad (\alpha \text{ known})$$

1(d)

$$L(\theta) = \frac{(\alpha\theta^3)^n}{n!} e^{-\alpha\theta^3} \prod_{i=1}^{n} \frac{x_i}{2\theta^3} e^{-x_i/\theta}$$

$$\ln L(\theta) = \cancel{3n\ln\theta} - \alpha\theta^3 - \cancel{3n\ln\theta} - \sum_{i=1}^{n} \frac{x_i}{\theta} + C$$

$$= -\alpha\theta^3 - \sum_{i=1}^{n} \frac{x_i}{\theta} + C$$

$$\frac{\partial \ln L}{\partial \theta} = -3\alpha\theta^2 + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i \overset{\text{set}}{=} 0$$

$$\Rightarrow \quad \hat{\theta} = \left(\frac{1}{3n} \sum_{i=1}^{n} x_i \right)^{1/4}$$

1e) $$E\left[a(n,\vec{x})\right] = \sum_{n=0}^{\infty} \int a(n,\vec{x})\, P(n,\vec{x})\, d\vec{x}$$

Joint probability for n & $\vec{x}$ is

$$P(n,\vec{x}) = f(\vec{x}\,|\,n)\, P(n)$$

$$\Rightarrow E\left[a(n,\vec{x})\right] = \sum_{n=0}^{\infty} P(n) \underbrace{\int a(n,\vec{x})\, f(\vec{x}\,|\,n)\, d\vec{x}}$$

$$= \sum_{n=0}^{\infty} P(n)\, E_{\vec{x}}\left[a(n,\vec{x})\,|\,n\right]$$

$$= E_n\left[E_{\vec{x}}\left[a(n,\vec{x})\,|\,n\right]\right]$$

1f) $$\frac{\partial^2 \ln L}{\partial \theta^2} = -6\alpha\theta - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i \quad \Big\}$$ function of both n and $\vec{x}$

$$E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = -6\alpha\theta - \frac{2}{\theta^3} E\left[\sum_{i=1}^{n} x_i\right]$$

use d) $$E\left[\sum_{i=1}^{n} x_i\right] = E_n\left[E_{\vec{x}}\left[\sum_{i=1}^{n} x_i \,\Big|\, n\right]\right] = E_n\left[\sum_{i=1}^{n} \underbrace{E[x_i]}_{= 3\theta}\right]$$

$$= E_n\left[3n\theta\right] = 3\underbrace{\nu}_{\alpha\theta^3}\theta = 3\alpha\theta^4$$

$$\Rightarrow E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = -6\alpha - \frac{2}{\theta^3}\,3\alpha\theta^4 = -12\alpha\theta$$

$$\Rightarrow V\left[\hat{\theta}\right] = \frac{1}{12\alpha\theta} = \frac{\theta^2}{12\nu}$$ compare to $\theta^2/3n$ for fixed n case.

use $\alpha = \nu/\theta^3$