

SDA - Discussion Session Notes - Week 5

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Problems sheet 2:

$$1) \quad f(x, y) = \frac{1}{\pi R^2} \quad x^2 + y^2 \leq R^2$$

$$\text{Define } u = \sqrt{x^2 + y^2} \quad u \geq 0$$

$$v = \tan^{-1} \frac{y}{x} \quad 0 \leq v \leq 2\pi$$

a) Find pdf $g(u, v)$

$$\text{Use inverse } x = u \cos v$$

$$y = u \sin v$$

Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos v & -u \sin v \\ \sin v & u \cos v \end{vmatrix}$$

$$= u \cos^2 v + u \sin^2 v = u$$

$$\Rightarrow g(u, v) = |J| f(x(u, v), y(u, v))$$

$$= \frac{u}{\pi R^2}, \quad 0 \leq u \leq R$$

 $g(u, v)$ is independent of v \Rightarrow joint pdf factorizes \Rightarrow u and v are independent.

1 b) marginal pdfs of u & v :

$$g_u(u) = \int g(u, v) dv$$

$$= \int_0^{2\pi} \frac{u}{\pi R^2} dv$$

$$= \frac{2u}{R^2}, \quad 0 \leq u \leq R$$

$$g_v(v) = \int g(u, v) du$$

$$= \int_0^R \frac{u}{\pi R^2} du$$

$$= \frac{1}{\pi R^2} \left. \frac{u^2}{2} \right|_0^R$$

$$= \frac{1}{2\pi}, \quad 0 \leq v \leq 2\pi$$

$$2) \quad E\left[c_0 + \sum_{i=1}^n c_i x_i\right] = \int (c_0 + \sum_{i=1}^n c_i x_i) f(\vec{x}) d\vec{x}$$

$$= c_0 \underbrace{\int f(\vec{x}) d\vec{x}}_1 + \sum_{i=1}^n c_i \underbrace{\int x_i f(\vec{x}) d\vec{x}}_{E[x_i]}$$

$$= c_0 + \sum_{i=1}^n c_i E[x_i]$$

$$V\left[c_0 + \sum_{i=1}^n c_i x_i\right] = E\left[\left(c_0 + \sum_{i=1}^n c_i x_i\right)^2\right] - \left(E\left[c_0 + \sum_{i=1}^n c_i x_i\right]\right)^2$$

$$= E\left[\left(c_0 + \sum_{i=1}^n c_i x_i\right)\left(c_0 + \sum_{j=1}^n c_j x_j\right)\right] - \left(c_0 + \sum_{i=1}^n c_i E[x_i]\right)\left(c_0 + \sum_{j=1}^n c_j E[x_j]\right)$$

$$= \cancel{c_0^2} + \cancel{2c_0 \sum_{i=1}^n c_i E[x_i]} + \sum_{i,j=1}^n c_i c_j E[x_i x_j]$$

$$- \cancel{c_0^2} - \cancel{2c_0 \sum_{i=1}^n c_i E[x_i]} - \sum_{i,j=1}^n c_i c_j E[x_i] E[x_j]$$

$$= \sum_{i,j=1}^n c_i c_j \left(E[x_i x_j] - E[x_i] E[x_j] \right)$$

$$= \sum_{i,j=1}^n c_i c_j \operatorname{cov}[x_i, x_j] \quad (\text{indep. of } c_0)$$

2 cont.)

If the x_i are uncorrelated,

$$\text{cov}[x_i, x_j] = \delta_{ij} \sigma_i^2$$

$$\Rightarrow V\left[c_0 + \sum_{i=1}^n c_i x_i\right] = \sum_{i,j=1}^n c_i c_j \text{cov}[x_i, x_j]$$

$$= \sum_{i,j=1}^n c_i c_j \delta_{ij} \sigma_i^2$$

$$= \sum_{i=1}^n c_i^2 \sigma_i^2$$

3) We are given

$$V[\alpha x + y] = \alpha^2 \sigma_x^2 + \sigma_y^2 + 2\alpha \sigma_x \sigma_y \rho \geq 0$$

$$\text{Let } \alpha = \frac{\sigma_y}{\sigma_x}$$

$$\Rightarrow \frac{\sigma_y^2}{\sigma_x^2} \sigma_x^2 + \sigma_y^2 + 2 \left(\frac{\sigma_y}{\sigma_x} \right) \rho \sigma_x \sigma_y \geq 0$$

$$\Rightarrow 2\sigma_y^2 + 2\rho\sigma_y^2 \geq 0 \Rightarrow \underline{\rho \geq -1}$$

$$\text{Let } \alpha = -\sigma_y / \sigma_x$$

$$\left(\frac{-\sigma_y}{\sigma_x} \right)^2 \sigma_x^2 + \sigma_y^2 - 2 \frac{\sigma_y}{\sigma_x} \rho \sigma_x \sigma_y \geq 0$$

$$\Rightarrow \sigma_y^2 + \sigma_y^2 - 2\rho\sigma_y^2 \geq 0 \Rightarrow \rho \leq 1$$

$$\Rightarrow \boxed{-1 \leq \rho \leq 1}$$