SDA Discussion notes week 7

Prob. Sheet 4 solutions:

1a) $X_N = \sum_{i=1}^{N} r_i$, r_i indep. & $\sim U[0,1]$

$$E[X_N] \equiv \mu_N = E\left[\sum_{i=1}^{N} r_i\right]$$

$$= \sum_{i=1}^{N} E[r_i] = \sum_{i=1}^{N} \frac{1}{2} = \frac{N}{2}$$

($E[r_i] = \frac{1}{2}$ from lect. notes)

$$V[X_N] = V\left[\sum_{i=1}^{N} r_i\right]$$

$$= \sum_{i=1}^{N} V[r_i] \quad \text{(since } r_i \text{ indep.)}$$

($V[r_i] = \frac{1}{12}$ from lecture notes)

$$= \frac{N}{12}$$

$$\Rightarrow \sigma_N = \sqrt{\frac{N}{12}}$$

$\Rightarrow$ By construction $y_N = \dfrac{X_N - \mu_N}{\sigma_N} = \sqrt{\dfrac{12}{N}}\left(\sum_{i=1}^{N} r_i - \dfrac{N}{2}\right)$

has $E[y_N] = 0$, $V[y_N] = 1$ ("standardized").

2) $f(x) = 4x^3 \quad , \quad 0 \le x \le 1$

2a) Transformation method

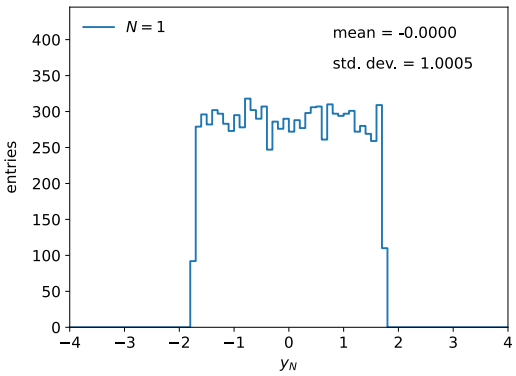
First find cumulative distribution

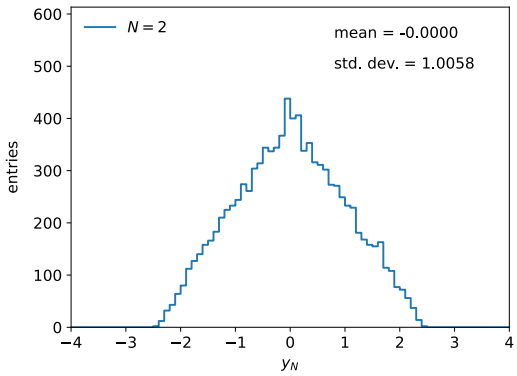
$$F(x) = \int_0^x 4x'^3 \, dx' = x^4$$

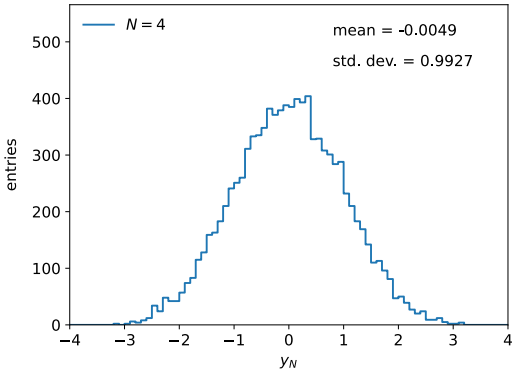
Set $F(x) = x^4 = r$

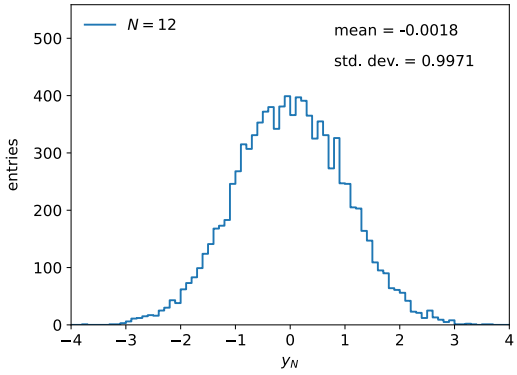
$$\Rightarrow \quad x(r) = r^{1/4}$$

1(b) [7 marks] (code for Exercises 1 and 2 at end). Histogram of y_N *for* N = 1, 2, 4, 12. The means and standard deviations are indicated on the plots. As shown, they are close to $\mu = 0$ and $\sigma = 1$, as must emerge by construction for the standardized variable y_N.









Exercise 2 We are given the pdf $f(x) = 4x^3$ for $0 < x < 1$.

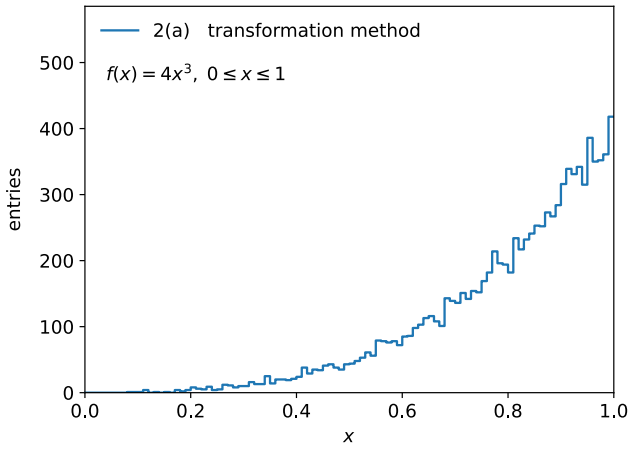
2(a) [4 marks] Find the transformation $x(r)$ to produce $x \sim f(x)$. First find the cumulative distribution

$$F(x) = \int_0^x 4x'^3\, dx' = x^4 \quad .$$

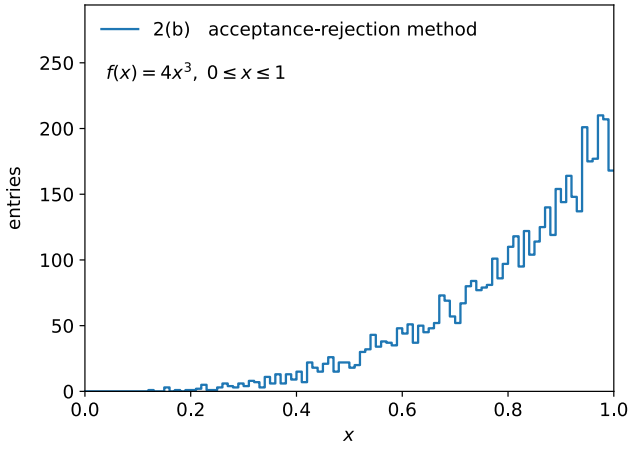
Set $F(x) = r$ and solving for *x* gives

$$x(r) = r^{1/4} \quad .$$

Histogram of values generated from the transformation method:



2(b) [4 marks] Histogram of values generated with the acceptance-rejection method (code at end):



Python code for questions 1 and 2:

```
# simpleMC.py -- simple Monte Carlo program to make histogram of uniformly
# distributed random values and plot
# G. Cowan, RHUL Physics, November 2021

import matplotlib
import matplotlib.pyplot as plt
import numpy as np

# Solution to problem sheet 4

# First make a simple routine to plot histograms
def plotHist (hist, bin_edges, xLabel, yLabel, plotLabel):
    matplotlib.rcParams.update({'font.size':12})     # set all font sizes
    xMin = bin_edges[0]
    xMax = bin_edges[len(bin_edges)-1]
    yMin = 0.
    yMax = np.max(hist)*1.4
    binLo, binHi = bin_edges[:-1], bin_edges[1:]
    xPlot = np.array([binLo, binHi]).T.flatten()
    yPlot = np.array([hist, hist]).T.flatten()
    fig, ax = plt.subplots(1,1)
    plt.gcf().subplots_adjust(bottom=0.15)
    plt.gcf().subplots_adjust(left=0.15)
    ax.set_xlim((xMin, xMax))
    ax.set_ylim((yMin, yMax))
    plt.xlabel(xLabel, labelpad=5)
    plt.ylabel(yLabel, labelpad=10)
    plt.plot(xPlot, yPlot, label=plotLabel)
    ax.legend(loc='upper left', frameon=False)
    plt.show()

# The function for yN returns an array of values
def yArr(N, numVal):
    rSum = np.zeros(numVal)
    for i in range (N):
        rSum += np.random.uniform(0., 1., numVal)
    return np.sqrt(12./N)*(rSum - N/2.)

numVal = 10000
nBins = 80
yMin = -4.
yMax = 4.

for N in [1, 2, 4, 12]:
    y = yArr(N, numVal)
    histLabel = '$N = $' + str(N)
    yHist, bin_edges = np.histogram(y, bins=nBins, range=(yMin, yMax))
    plotHist(yHist, bin_edges, r'$y_{N}$', r'entries', histLabel)
    plt.figtext(0.6, 0.81, f'mean = {y.mean():.4f}')
    plt.figtext(0.6, 0.74, f'std. dev. = {y.std():.4f}')


# 2(a) Generate random values according to f(x) = 4x^3 (0 < x < 1)
# with transformation method; here x = r**(1./4.)
numVal = 10000
nBins = 100
rData = np.random.uniform(0., 1., numVal)
xMin=0.
xMax=1.
xData = pow(rData, 0.25)
xHist, bin_edges = np.histogram(xData, bins=nBins, range=(xMin, xMax))
```

```
plotHist(xHist, bin_edges, r'$x$', r'entries', r'2(a)   transformation method')
plt.figtext(0.18, 0.74, r'$f(x) = 4x^{3}, \; 0 \leq x \leq 1$')


# 2(b) Repeat using the acceptance rejection method
numVal = 20000
nBins = 100

def f(x):
    return 4. * x**3

xMin = 0.
xMax = 1.
fMax = 4.
r1 = np.random.uniform(0., 1., numVal)
x = xMin + r1*(xMax - xMin)
r2 = np.random.uniform(0., 1., numVal)
u = fMax*r2
xData = x[u<f(x)]
nBins = 100
xHist, bin_edges = np.histogram(xData, bins=nBins, range=(xMin, xMax))
plotHist(xHist, bin_edges, r'$x$', r'entries', r'2(b)   acceptance-rejection
method')
plt.figtext(0.18, 0.74, r'$f(x) = 4x^{3}, \; 0 \leq x \leq 1$')
```

C++ code for questions 1 and 2:

```
// Solution to Computing and Statistics Monte Carlo Problem
// Glen Cowan, RHUL, Physics, November 2021

#include <iostream>
#include <cmath>
#include <TH1D.h>
#include <TFile.h>
#include <TRandom3.h>

using namespace std;

// Define the function yN

double y(int N, TRandom3* ran){
  double rSum = 0.;
  for(int i=0; i<N; i++){
    rSum += ran->Rndm();
  }
  return sqrt(12./N)*(rSum - N/2.);
}

int main(){

// Open output file (apparently needs to be done before booking)

  TFile* file = new TFile("simpleMC.root", "recreate");

// Book histograms

  TH1D* h_y1 = new TH1D("h_y1", "N = 1", 100, -4., 4.);
  TH1D* h_y2 = new TH1D("h_y2", "N = 2", 100, -4., 4.);
  TH1D* h_y4 = new TH1D("h_y4", "N = 4", 100, -4., 4.);
  TH1D* h_y12 = new TH1D("h_y12", "N = 12", 100, -4., 4.);
  TH1D* h_trans = new TH1D("h_trans", "x", 100, 0., 1.);
  TH1D* h_accrej = new TH1D("h_accrej", "x", 100, 0., 1.);

// Create a TRandom3 object to generate random numbers
```

```
  int seed = 12345;
  TRandom3* ran = new TRandom3(seed);

// Generate values and fill histograms

  const int numValues = 10000;
  for (int i=0; i<numValues; ++i){
    double y1 = y(1, ran);
    double y2 = y(2, ran);
    double y4 = y(4, ran);
    double y12 = y(12, ran);
    h_y1->Fill(y1);
    h_y2->Fill(y2);
    h_y4->Fill(y4);
    h_y12->Fill(y12);
  }

// Generate random values according to f(x) = 4x^3 (0 < x < 1)
// with transformation method; here x = r^0.25.

  for (int i=0; i<numValues; i++){
    double r = ran->Rndm();
    double x = pow(r, 0.25);
    h_trans->Fill(x);
  }

// and again using acceptance rejection method

  const double xMin = 0.;
  const double xMax = 1.;
  const double fMax = 4.;
  int numAccepted = 0;
  while ( numAccepted < numValues ){
    double x = (xMax - xMin)*ran->Rndm() + xMin;
    double u = fMax * ran->Rndm();
    double f = 4.*pow(x,3);
    if ( u < f ){
      numAccepted++;
      h_accrej->Fill(x);
    }
  }

// Store all histograms in the output file and close up

  file->Write();
  file->Close();

  return 0;
}
```

3) $\vec{x} \sim \text{Gauss}(\vec{\mu}_k, V)$, $k = 0, 1$

($\vec{x} = (x_1, \ldots, x_n)$; V = covariance matrix)

i.e. $f(\vec{x}|\vec{\mu}_k) = \frac{1}{(2\pi)^{n/2}|V|^{1/2}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu}_k)^T V^{-1}(\vec{x}-\vec{\mu}_k)\right]$

Test statistic $t(x) = \ln \frac{f(\vec{x}|\vec{\mu}_1)}{f(\vec{x}|\vec{\mu}_0)}$

$$\ln \frac{f(\vec{x}|\vec{\mu}_1)}{f(\vec{x}|\vec{\mu}_0)} = -\frac{1}{2}\left[(\vec{x}-\vec{\mu}_1)^T V^{-1}(\vec{x}-\vec{\mu}_1) - (\vec{x}-\vec{\mu}_0)^T V^{-1}(\vec{x}-\vec{\mu}_0)\right]$$

$$= -\frac{1}{2}\left[\vec{x}^T V^{-1}\vec{x} - \vec{\mu}_1^T V^{-1}\vec{x} - \vec{x}^T V^{-1}\vec{\mu}_1 + \vec{\mu}_1^T V^{-1}\vec{\mu}_1 - \vec{x}^T V^{-1}\vec{x} + \vec{\mu}_0^T V^{-1}\vec{x} + \underbrace{\vec{x}^T V^{-1}\vec{\mu}_0}_{= \vec{\mu}_0^T V^{-1}\vec{x}} - \vec{\mu}_0^T V^{-1}\vec{\mu}_0\right]$$

since scalar $() = ()^T$ and $V^{-1} = (V^{-1})^T$

$$= \underbrace{-\frac{1}{2}\left[\vec{\mu}_1 V^{-1}\vec{\mu}_1 - \vec{\mu}_0^T V^{-1}\vec{\mu}_0\right]}_{w_0} + \underbrace{(\vec{\mu}_1 - \vec{\mu}_0)^T V^{-1}}_{\vec{w}^T}\vec{x}$$

$$= w_0 + \vec{w}^T\vec{x}$$

$$\vec{w} = \left[(\vec{\mu}_1 - \vec{\mu}_0)^T \underbrace{V^{-1}}_{= V^{-1}}\right]^T = (V^{-1})^T(\vec{\mu}_1 - \vec{\mu}_0)$$

If $W = V + V$, $W^{-1} = \frac{1}{2}V^{-1}$

$$\Rightarrow \vec{w} = 2W^{-1}(\vec{\mu}_1 - \vec{\mu}_0)$$

Example of MLE

Consider $f(x) = (1+\theta)x^{\theta}$, $0 \le x \le 1$

with i.i.d. sample $x_1, \ldots, x_n$

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} (1+\theta) x_i^{\theta}$$

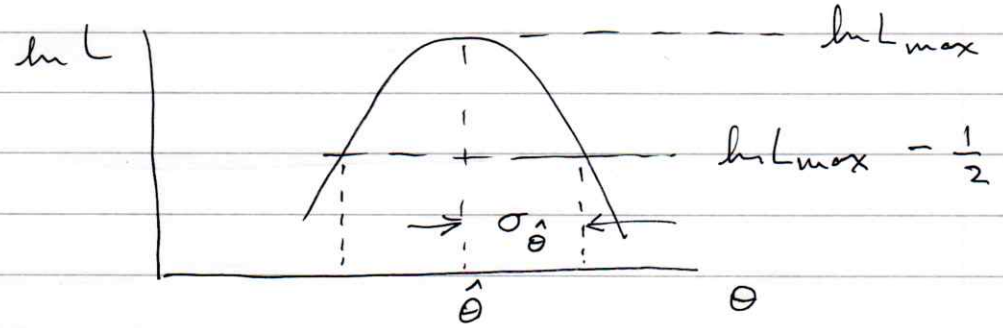
$$\Rightarrow \ln L(\theta) = \sum_{i=1}^{n} \left[\ln(1+\theta) + \theta \ln x_i \right]$$

To find MLE,

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{1+\theta} + \sum_{i=1}^{n} \ln x_i \overset{\text{set}}{=} 0$$

$$\Rightarrow \hat{\theta} = -1 - \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

For variance (graphical method)



$\sigma_{\hat{\theta}}$ found by moving θ away from $\hat{\theta}$ until $\ln L$ decreases by $1/2$ from $\ln L_{max}$.

Variance from asymptotic properties

Assume large n, $\underbrace{\frac{\partial b}{\partial \theta}} \to \approx 0$

$$V[\hat{\theta}] \approx MVB = -\frac{\left(1+\frac{\partial b}{\partial \theta}\right)^2}{E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right]}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n}{(1+\theta)^2} \quad \text{or indep. of data } x_i$$

$$-E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = \frac{n}{(1+\theta)^2}$$

$$\Rightarrow V[\hat{\theta}] \approx \frac{(1+\theta)^2}{n}$$

$$\sigma_{\hat{\theta}} \approx \frac{1+\theta}{\sqrt{n}}$$

MLE Example (cont.)

Now suppose problem is parametrized using $\lambda = \exp\left[\frac{-1}{1+\theta}\right]$ ⟵ find MLE of λ.

Use $\hat{\lambda} = \lambda(\hat{\theta})$

$$\Rightarrow \hat{\lambda} = \exp\left[-\frac{1}{1+\left(-1-\frac{n}{\sum_{i=1}^{n} \ln x_i}\right)}\right]$$

$$= \exp\left[\frac{1}{n}\sum_{i=1}^{n} \ln x_i\right] = \exp\left[\sum_{i} \ln x_i^{1/n}\right]$$

$$= \prod_{i=1}^{n} x_i^{1/n}$$

Or the hard way: $\theta = -1 - \frac{1}{\ln\lambda}$

$$f(x;\lambda) = -\frac{1}{\ln\lambda} x^{\left(-1-\frac{1}{\ln\lambda}\right)}$$

$$L(\lambda) = \prod_{i=1}^{n} f(x_i;\lambda)$$

$\rightarrow \ln\lambda \longrightarrow \frac{\partial \ln L}{\partial \lambda} = 0 \rightarrow$ same $\hat{\lambda}$ as above.