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SDA Problem Sheet 6

1) $n \sim \text{Poisson}(s+b)$, $b = 3.7$

$n_{obs} = 15$

a) $p_0 = P(n \geq n_{obs} \mid s = 0, b = 3.7)$

$$= \sum_{n=n_{obs}}^{\infty} \frac{b^n}{n!} e^{-b}$$

$$= 1 - \sum_{n=0}^{n_{obs}-1} \frac{b^n}{n!} e^{-b}$$

upper limit of sum

$$= F_{\chi^2}(2b; 2n_{obs}) \qquad [n_{dof} = 2(m+1) = 2n_{obs}]$$

= scipy.stats.chi2.cdf(2*3.7, 2*15)

= 1 - TMath::Prob(2*3.7, 2*15)

$= \underline{8.2 \times 10^{-6}}$

b) $Z_0 = \Phi^{-1}(1 - p_0)$

= scipy.stats.norm.ppf(1 - p₀)

= TMath::NormQuantile(1 - p₀)

$= \underline{4.3}$

2) $x \sim \text{Gauss}(\mu, \sigma^2)$

i.i.d. sample $x_1, \ldots, x_N$

2a) $$L(\mu, \sigma^2) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_i-\mu)^2/2\sigma^2}$$

$$\Rightarrow \ln L(\mu, \sigma^2) = -\frac{1}{2}\sum_{i=1}^{N} \frac{(x_i-\mu)^2}{\sigma^2} - \frac{N}{2}\ln\sigma^2 + C$$

2b) $$\frac{\partial \ln L}{\partial \mu} = \sum_{i=1}^{N} \frac{x_i - \mu}{\sigma^2} \overset{\text{set}}{=} 0 \qquad (1)$$

$$\frac{\partial \ln L}{\partial \sigma^2} = \frac{1}{2}\sum_{i=1}^{N} \frac{(x_i-\mu)^2}{(\sigma^2)^2} - \frac{N}{2\sigma^2} \overset{\text{set}}{=} 0 \qquad (2)$$

From (1) $$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

From (2) $$\widehat{\sigma^2} = \frac{1}{N}\sum_{i=1}^{N} (x_i - \hat{\mu})^2$$

2c) Fisher information matrix

$$I_{ij} = -E\left[\frac{\partial^2 \ln L}{\partial\theta_i\,\partial\theta_j}\right] \qquad \theta_1 = \mu, \quad \theta_2 = \sigma^2$$

$$\frac{\partial^2 \ln L}{\partial \mu^2} = -\frac{N}{\sigma^2}$$

$$\frac{\partial^2 \ln L}{\partial (\sigma^2)^2} = \frac{N}{2\sigma^4} - \sum_{i=1}^{N} \frac{(x_i-\mu)^2}{\sigma^6}$$

$$\frac{\partial^2 \ln L}{\partial \mu\,\partial \sigma^2} = -\frac{1}{\sigma^4}\sum_{i=1}^{N}(x_i-\mu)$$

$$E\left[\frac{\partial^2 \ln L}{\partial \mu^2}\right] = -\frac{N}{\sigma^2}$$

$$E\left[\frac{\partial^2 \ln L}{\partial (\sigma^2)^2}\right] = \frac{N}{2\sigma^4} - \sum_{i=1}^{N}\frac{\overbrace{E\left[(x_i-\mu)^2\right]}^{\sigma^2}}{\sigma^6} = -\frac{N}{2\sigma^4}$$

$$E\left[\frac{\partial^2 \ln L}{\partial \mu\,\partial \sigma^2}\right] = -\frac{1}{\sigma^4}\sum_{i=1}^{N}\underbrace{E\left[x_i-\mu\right]}_{=0} = 0$$

$$\Rightarrow \quad I = \begin{pmatrix} \frac{N}{\sigma^2} & 0 \\ 0 & \frac{N}{2\sigma^4} \end{pmatrix}$$

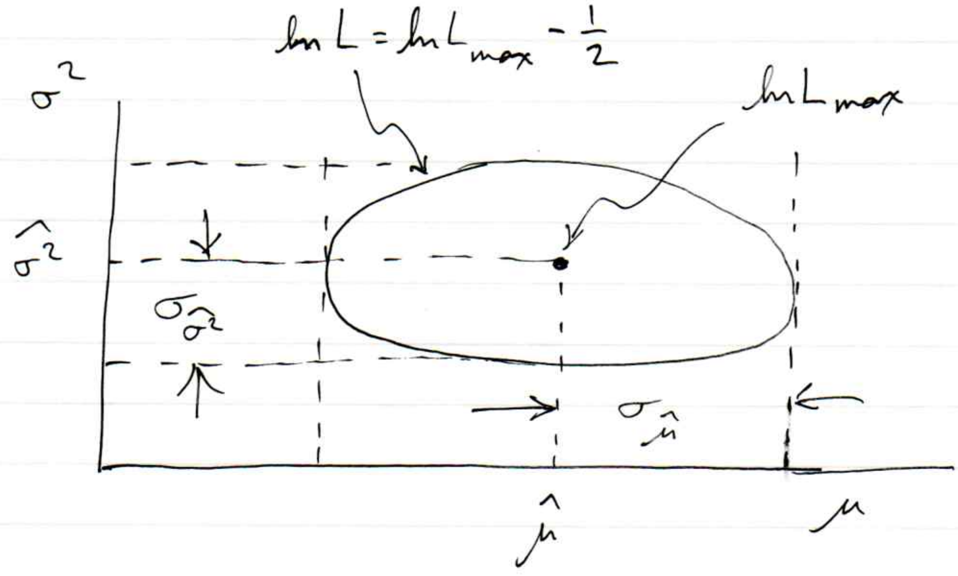
2d) Use $V^{-1} = I$

assumes zero bias, info inequality → equality

Since I diagonal, inverse by inspection

$$V = \begin{pmatrix} \frac{\sigma^2}{N} & 0 \\ 0 & \frac{2\sigma^4}{N} \end{pmatrix} = \begin{pmatrix} V[\hat{\mu}] & \mathrm{cov}[\hat{\mu}, \widehat{\sigma^2}] \\ \mathrm{cov}[\hat{\mu}, \widehat{\sigma^2}] & V[\widehat{\sigma^2}] \end{pmatrix}$$

2e)



No tilt, corresponds to $\mathrm{cov}[\hat{\mu}, \widehat{\sigma^2}] = 0$