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Note on the Exponential Distribution

Suppose an unstable elementary particle has no "internal clock" and thus no record of how long it has existed. It just exists, and at any time t it has a certain probability of decaying in the interval between t and $t + \delta t$. Let us suppose that for very small δt, this probability is proportional to δt,

$$P(\text{decay in } [t, t + \delta t]) = \lambda \, \delta t \; , \tag{1}$$

and suppose furthermore that the proportionality constant λ is independent of the time.

Suppose the particle exists at time $t = 0$ and we want to know the probability that it will decay before some time t. Let us subdivide the interval from 0 to t into a large number N of small parts each of duration δt, i.e., $t = N \, \delta t$. For the particle to live until t, it must survive each of the individual subintervals, and the probability to survive one of those is $1 - \lambda \, \delta t$. The probability to not decay before t is therefore the product of the individual probabilities, i.e.,

$$\begin{aligned} P(\text{no decay in } [0, t]) &= \prod_{n=1}^{N} P(\text{no decay in } [(n-1)\delta t, n \, \delta t]) \\ &= (1 - \lambda \, \delta t)^N \\ &= \left(1 - \frac{\lambda t}{N}\right)^N \; . \end{aligned} \tag{2}$$

In the limit where N goes to infinity and thus $\delta t = t/N \to dt$ becomes infinitessimally small, the product above becomes an exponential function, and therefore

$$P(\text{no decay in } [0, t]) = e^{-\lambda t} \; . \tag{3}$$

It is useful to relate this probability to the corresponding *probability density function* or pdf, $p(t)$. This is defined such that the probability for the decay to occur in the interval $[t, t + dt]$ is

$$P(\text{decay occurs between } t \text{ and } t + dt) = p(t) dt \; . \tag{4}$$

The probability for the decay to occur between any two limits is found simply by integrating the pdf, and therefore

$$P(\text{decay occurs in } [0, t]) = \int_0^t p(t') \, dt' \; . \tag{5}$$

If we wait long enough, an unstable particle will eventually decay with probability of one, and so we have $\int_0^\infty p(t) \, dt = 1$.

The quantity $P(\text{decay occurs in } [0,t]) \equiv P(t)$ is called the cumulative exponential distribution. From Eq. (3) we have

$$P(t) = 1 - P(\text{no decay in } [0,t]) = 1 - e^{-\lambda t} = \int_0^t p(t')\, dt' \; . \tag{6}$$

By differentiating both sides of Eq. (5) and solving for $p(t)$ we find the exponential pdf

$$p(t) = \frac{dP}{dt} = \lambda e^{-\lambda t} \; . \tag{7}$$

The mean value of the decay time t is found by the usual relation (see, e.g., Ref. [1]),

$$\langle t \rangle = \int_0^t t e^{-\lambda t}\, dt = \frac{1}{\lambda} \tag{8}$$

We can define $\tau = 1/\lambda$ as the mean lifetime, and the exponential pdf is often seen in the equivalent form

$$p(t) = \frac{1}{\tau} e^{-t/\tau} \; . \tag{9}$$

References

[1] G. Cowan, *Statistical Data Analysis*, Oxford University Press, 1998.