Statistical Data Analysis
Problem Sheet 9
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1(a) [3 marks] The log-likelihood is obtained directly from the pdf for t,

$$\ln L(\tau) = \ln f(t|\tau) = -\ln \tau - \frac{t}{\tau} \ .$$

Setting the derivative of $\ln L(\tau)$ to zero,

$$\frac{\partial \ln L}{\partial \tau} = -\frac{1}{\tau} + \frac{t}{\tau^2} = 0 \ ,$$

and solving for τ gives $\hat{\tau} = t$.

1(b-i) [3 marks] For a test of size α, the critical region, $t \geq t_{\rm cut}$, is determined by the requirement

$$\alpha = P(t \geq t_{\rm cut}|\tau) = \int_{t_{\rm cut}}^{\infty} \frac{1}{\tau} e^{-t/\tau} \, dt = e^{-t_{\rm cut}/\tau} \ .$$

Solving for $t_{\rm cut}$ therefore gives $t_{\rm cut} = -\tau \ln \alpha$.

1(b-ii) [2 marks] Taking larger t values to constitute increasing incompatibility with τ, the p-value is the probability to observe a value greater than or equal to t, i.e.,

$$p = \int_{t}^{\infty} \frac{1}{\tau} e^{-t'/\tau} \, dt' = e^{-t/\tau} \ .$$

1(b-iii) [2 marks] The lower limit at confidence level $1 - \alpha$ is the value of τ for which $p = \alpha$, i.e.,

$$\tau_{\rm lo} = -\frac{t}{\ln \alpha} = -\frac{1\,{\rm s}}{\ln(0.05)} = 0.33\,{\rm s} \ .$$

1(c) [4 marks] The Jeffreys prior is defined by

$$\pi(\tau) \propto \sqrt{I(\tau)} \ ,$$

where

$$I(\tau) = -E\left[\frac{\partial^2 \ln L}{\partial \tau^2}\right]$$

is the Fisher information. The required ingredients are

$$
\begin{aligned}
\frac{\partial^2 \ln L}{\partial \tau^2} &= \frac{1}{\tau^2} - \frac{2t}{\tau^3} \,, \\
-E\left[\frac{\partial^2 \ln L}{\partial \tau^2}\right] &= -\frac{1}{\tau^2} + \frac{2E[t]}{\tau^3} = \frac{1}{\tau^2} \,,
\end{aligned}
$$

where to arrive at the final equality we used the expectation value $E[t] = \tau$. The Jeffreys prior is therefore

$$
\pi(\tau) \propto \frac{1}{\tau} \qquad (\tau \geq 0).
$$

1(d) [**3 marks**] From Bayes' theorem we find the posterior pdf up to a normalization constant C,

$$
\begin{aligned}
p(\tau|t) &\propto f(t|\tau)\pi(\tau) \\
&= C\frac{1}{\tau}e^{-t/\tau}\frac{1}{\tau} \,.
\end{aligned}
$$

To determine the constant C we require

$$
C \int_0^\infty \frac{1}{\tau^2} e^{-t/\tau} \, d\tau = 1 \,.
$$

To calculate the integral let $u = t/\tau$, $du = (t/u^2)du$, which gives

$$
C \int_\infty^0 \left(\frac{u}{t}\right)^2 e^{-u} \left(\frac{-t}{u^2}\right) du = Ct^{-1} = 1 \,,
$$

and therefore $C = t$. The normalized posterior pdf is therefore

$$
p(\tau|t) = \frac{t}{\tau^2} e^{-t/\tau} \,.
$$

1(e) [**3 marks**] The posterior mode is found from

$$
\frac{\partial p(\tau|t)}{\partial \tau} = \frac{t}{\tau^2} e^{-t/\tau} \left(\frac{t}{\tau^2}\right) + e^{-t/\tau} \left(\frac{-2t}{\tau^3}\right) = 0 \,.
$$

Solving for τ gives the mode,

$$
\tau_{\text{mode}} = \frac{t}{2} \,.
$$

From Bayes' theorem the posterior is $p(\tau|t) \propto L(\tau)\pi(\tau)$, so if the prior $\pi(\tau)$ is a constant, then the mode of $p(\tau|t)$ is at the same value as the maximum of the likelihood $L(\tau)$, i.e., the posterior mode is then equal to the ML estimator.

The Jeffreys prior, however, is $\pi(\tau) \propto 1/\tau$, so a priori lower values of τ are favoured, and this feature is inherited by the posterior pdf. Therefore the posterior mode is at a lower value ($\tau_{\text{mode}} = t/2$) when using the Jeffreys prior compared to the ML estimator $\hat{\tau}_{\text{ML}} = t$.