Statistical Data Analysis
Stat 4: confidence intervals, limits, discovery

London Postgraduate Lectures on Particle Physics;

University of London MSci course PH4515

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Interval estimation — introduction

In addition to a 'point estimate' of a parameter we should report an interval reflecting its statistical uncertainty.

Desirable properties of such an interval may include:

communicate objectively the result of the experiment;
have a given probability of containing the true parameter;
provide information needed to draw conclusions about the parameter possibly incorporating stated prior beliefs.

Often use +/− the estimated standard deviation of the estimator.
In some cases, however, this is not adequate:

estimate near a physical boundary,
e.g., an observed event rate consistent with zero.

We will look briefly at Frequentist and Bayesian intervals.

Frequentist confidence intervals

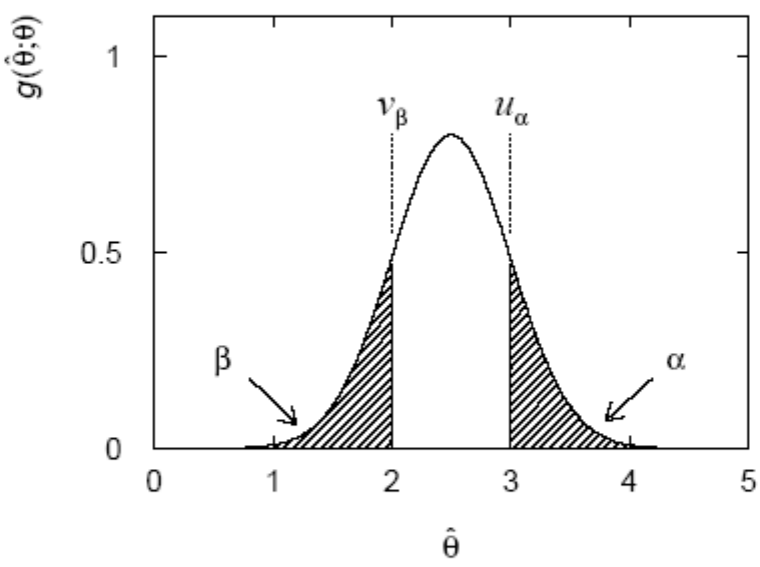
Consider an estimator $\hat{\theta}$ for a parameter θ and an estimate $\hat{\theta}_{\rm obs}$.

We also need for all possible θ its sampling distribution $g(\hat{\theta};\theta)$.

Specify upper and lower tail probabilities, e.g., $\alpha = 0.05$, $\beta = 0.05$, then find functions $u_\alpha(\theta)$ and $v_\beta(\theta)$ such that:

$$\alpha = P(\hat{\theta} \geq u_\alpha(\theta)) = \int_{u_\alpha(\theta)}^{\infty} g(\hat{\theta};\theta)\, d\hat{\theta}$$

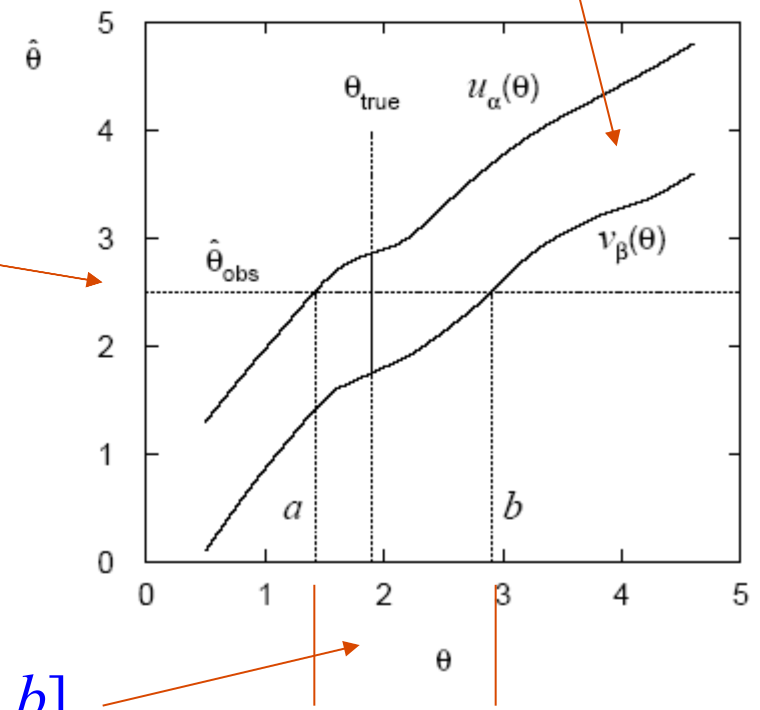
$$\beta = P(\hat{\theta} \leq v_\beta(\theta)) = \int_{-\infty}^{v_\beta(\theta)} g(\hat{\theta};\theta)\, d\hat{\theta}$$



Confidence interval from the confidence belt

The region between $u_\alpha(\theta)$ and $v_\beta(\theta)$ is called the confidence belt.

Find points where observed estimate intersects the confidence belt.



This gives the confidence interval $[a, b]$

Confidence level $= 1 - \alpha - \beta =$ probability for the interval to cover true value of the parameter (holds for any possible true θ).

Confidence intervals by inverting a test

Confidence intervals for a parameter θ can be found by defining a test of the hypothesized value θ (do this for all θ):

Specify values of the data that are 'disfavoured' by θ (critical region) such that P(data in critical region) $\leq \gamma$ for a prespecified γ, e.g., 0.05 or 0.1.

If data observed in the critical region, reject the value θ.

Now invert the test to define a confidence interval as:

set of θ values that would not be rejected in a test of size γ (confidence level is $1 - \gamma$).

The interval will cover the true value of θ with probability $\geq 1 - \gamma$.

Equivalent to confidence belt construction; confidence belt is acceptance region of a test.

Relation between confidence interval and p-value

Equivalently we can consider a significance test for each hypothesized value of θ, resulting in a p-value, p_θ.

If $p_\theta < \gamma$, then we reject θ.

The confidence interval at CL = $1 - \gamma$ consists of those values of θ that are not rejected.

E.g. an upper limit on θ is the greatest value for which $p_\theta \geq \gamma$.

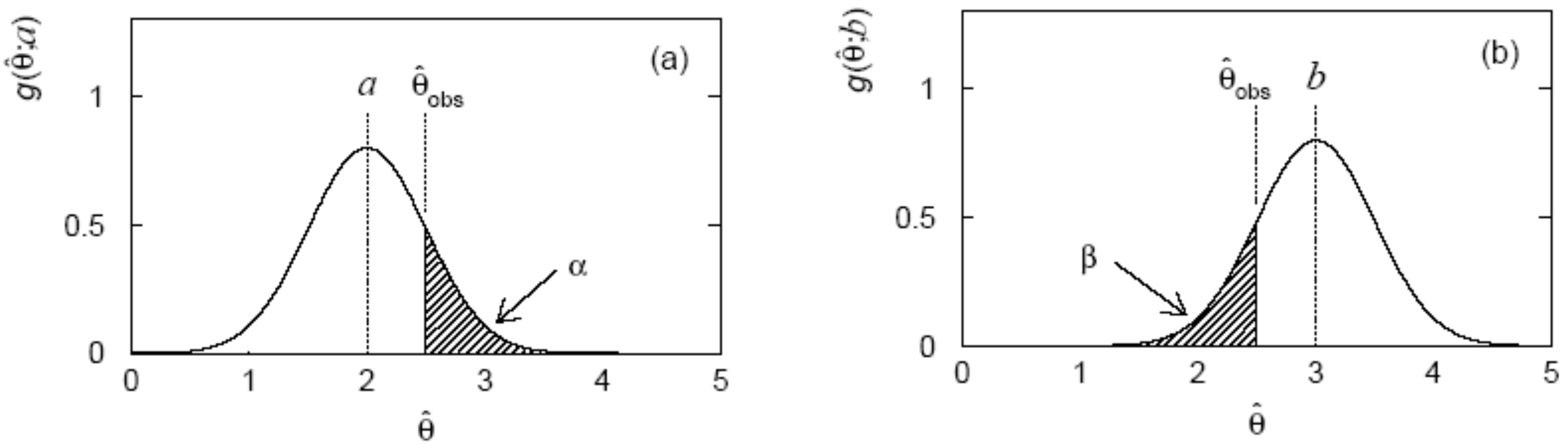
In practice find by setting $p_\theta = \gamma$ and solve for θ.

Confidence intervals in practice

The recipe to find the interval [a, b] boils down to solving

$$\alpha = \int_{u_\alpha(\theta)}^{\infty} g(\hat{\theta};\theta)\,d\hat{\theta} = \int_{\hat{\theta}_{\rm obs}}^{\infty} g(\hat{\theta};a)\,d\hat{\theta}\,,$$

$$\beta = \int_{-\infty}^{v_\beta(\theta)} g(\hat{\theta};\theta)\,d\hat{\theta} = \int_{-\infty}^{\hat{\theta}_{\rm obs}} g(\hat{\theta};b)\,d\hat{\theta}\,.$$



→ a is hypothetical value of θ such that $P(\hat{\theta} > \hat{\theta}_{\rm obs}) = \alpha\,.$

→ b is hypothetical value of θ such that $P(\hat{\theta} < \hat{\theta}_{\rm obs}) = \beta\,.$

Meaning of a confidence interval

N.B. the interval is random, the true θ is an unknown constant.

Often report interval $[a, b]$ as $\hat{\theta}^{+d}_{-c}$, i.e. $c = \hat{\theta} - a$, $d = b - \hat{\theta}$.

So what does $\hat{\theta} = 80.25^{+0.31}_{-0.25}$ mean? It does not mean:

$P(80.00 < \theta < 80.56) = 1 - \alpha - \beta$, but rather:

repeat the experiment many times with same sample size,

construct interval according to same prescription each time,

in $1 - \alpha - \beta$ of experiments, interval will cover θ.

Central vs. one-sided confidence intervals

Sometimes only specify α or β, $\rightarrow$ one-sided interval (limit)

Often take $\alpha = \beta = \dfrac{\gamma}{2}$ $\rightarrow$ coverage probability $= 1 - \gamma$

$\rightarrow$ central confidence interval

N.B. 'central' confidence interval does not mean the interval

is symmetric about $\hat{\theta}$, but only that $\alpha = \beta$.

The HEP error 'convention': 68.3% central confidence interval.

Approximate confidence intervals/regions from the likelihood function

Suppose we test parameter value(s) $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ using the ratio

$$\lambda(\boldsymbol{\theta}) = \frac{L(\boldsymbol{\theta})}{L(\hat{\boldsymbol{\theta}})} \qquad\qquad 0 \leq \lambda(\boldsymbol{\theta}) \leq 1$$

Lower $\lambda(\boldsymbol{\theta})$ means worse agreement between data and hypothesized $\boldsymbol{\theta}$. Equivalently, usually define

$$t_{\boldsymbol{\theta}} = -2 \ln \lambda(\boldsymbol{\theta})$$

so higher $t_{\boldsymbol{\theta}}$ means worse agreement between $\boldsymbol{\theta}$ and the data.

p-value of $\boldsymbol{\theta}$ therefore $\qquad p_{\boldsymbol{\theta}} = \int_{t_{\boldsymbol{\theta},\mathrm{obs}}}^{\infty} f(t_{\boldsymbol{\theta}}|\boldsymbol{\theta})\, dt_{\boldsymbol{\theta}}$

need pdf

Confidence region from Wilks' theorem

Wilks' theorem says (in large-sample limit and providing certain conditions hold...)

$$f(t_{\boldsymbol{\theta}}|\boldsymbol{\theta}) \sim \chi^2_n$$

chi-square dist. with # d.o.f. = # of components in $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$.

Assuming this holds, the *p*-value is

$$p_{\boldsymbol{\theta}} = 1 - F_{\chi^2_n}(t_{\boldsymbol{\theta}})$$

To find boundary of confidence region set $p_{\boldsymbol{\theta}} = \alpha$ and solve for $t_{\boldsymbol{\theta}}$:

$$t_{\boldsymbol{\theta}} = F^{-1}_{\chi^2_n}(1 - \alpha)$$

Recall also

$$t_{\theta} = -2 \ln \frac{L(\theta)}{L(\hat{\theta})}$$

Confidence region from Wilks' theorem (cont.)

i.e., boundary of confidence region in $\boldsymbol{\theta}$ space is where

$$\ln L(\boldsymbol{\theta}) = \ln L(\hat{\boldsymbol{\theta}}) - \tfrac{1}{2} F^{-1}_{\chi^2_n}(1-\alpha)$$

For example, for $1 - \alpha = 68.3\%$ and $n = 1$ parameter,

$$F^{-1}_{\chi^2_1}(0.683) = 1$$

and so the 68.3% confidence level interval is determined by

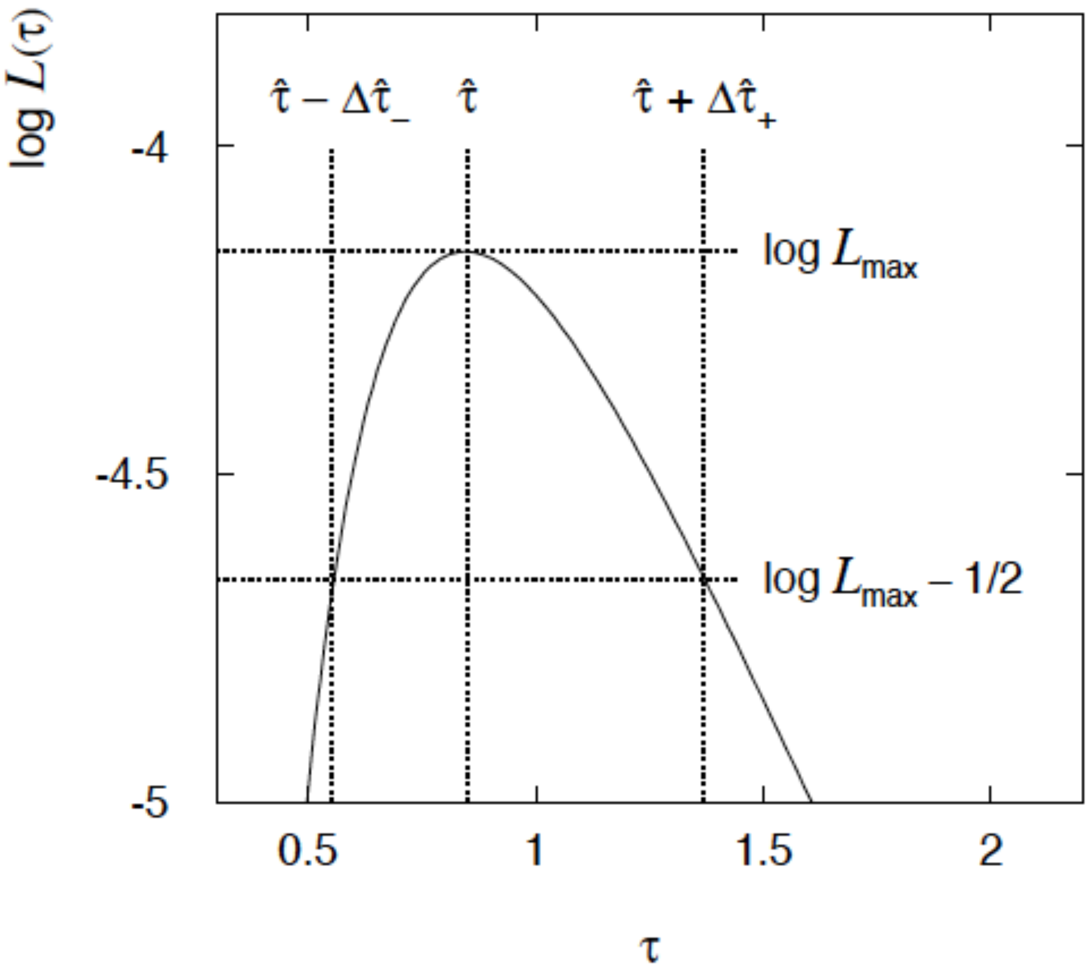
$$\ln L(\theta) = \ln L(\hat{\theta}) - \frac{1}{2}$$

Same as recipe for finding the estimator's standard deviation, i.e.,

$[\hat{\theta} - \sigma_{\hat{\theta}}, \hat{\theta} + \sigma_{\hat{\theta}}]$ is a 68.3% CL confidence interval.

Example of interval from ln $L(\theta)$

For n=1 parameter, CL = 0.683, $Q_\alpha = 1$.



Our exponential example, now with only n = 5 events.

Can report ML estimate with approx. confidence interval from ln L_{max} – 1/2 as "asymmetric error bar":

$$\hat{\tau} = 0.85^{+0.52}_{-0.30}$$

Multiparameter case

For increasing number of parameters, CL = 1 – α decreases for confidence region determined by a given

$$Q_\alpha = F^{-1}_{\chi^2_n}(1-\alpha)$$

Q_α	$1-\alpha$				
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
1.0	0.683	0.393	0.199	0.090	0.037
2.0	0.843	0.632	0.428	0.264	0.151
4.0	0.954	0.865	0.739	0.594	0.451
9.0	0.997	0.989	0.971	0.939	0.891

Multiparameter case (cont.)

Equivalently, Q_α increases with n for a given CL = $1 - \alpha$.

$1-\alpha$	Q_α				
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.683	1.00	2.30	3.53	4.72	5.89
0.90	2.71	4.61	6.25	7.78	9.24
0.95	3.84	5.99	7.82	9.49	11.1
0.99	6.63	9.21	11.3	13.3	15.1

Ingredients for a test / interval

Note that these confidence intervals can be found using only the likelihood function evaluated with the observed data. This is because the statistic

$$t_\theta = -2 \ln \frac{L(\theta)}{L(\hat{\theta})}$$

approaches a well-defined distribution independent of the distribution of the data in the large sample limit.

For finite samples, however, the resulting intervals are approximate.

In general to carry out a test we need to know the distribution of the test statistic $t(x)$, and this means we need the full model $P(x|\boldsymbol{\theta})$.

Frequentist upper limit on Poisson parameter

Consider again the case of observing $n \sim \text{Poisson}(s + b)$.

Suppose $b = 4.5$, $n_{\text{obs}} = 5$. Find upper limit on s at 95% CL.

Relevant alternative is $s = 0$ (critical region at low n)

p-value of hypothesized s is $P(n \leq n_{\text{obs}}; s, b)$

Upper limit s_{up} at $\text{CL} = 1 - \alpha$ found from

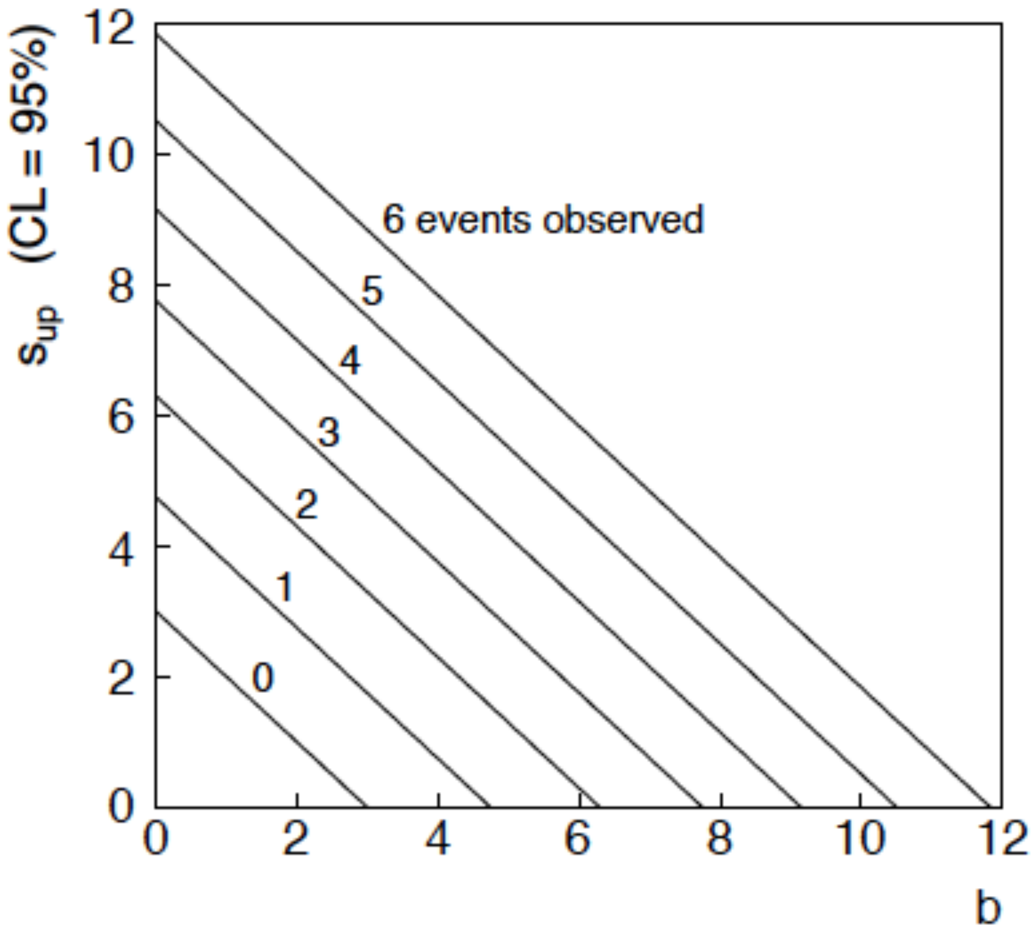
$$\alpha = P(n \leq n_{\text{obs}}; s_{\text{up}}, b) = \sum_{n=0}^{n_{\text{obs}}} \frac{(s_{\text{up}} + b)^n}{n!} e^{-(s_{\text{up}}+b)}$$

$$s_{\text{up}} = \frac{1}{2} F_{\chi^2}^{-1}(1 - \alpha; 2(n_{\text{obs}} + 1)) - b$$

$$= \frac{1}{2} F_{\chi^2}^{-1}(0.95; 2(5 + 1)) - 4.5 = 6.0$$

$n \sim$ Poisson($s+b$): frequentist upper limit on s

For low fluctuation of n formula can give negative result for s_{up}; i.e. confidence interval is empty.



Limits near a physical boundary

Suppose e.g. $b = 2.5$ and we observe $n = 0$.

If we choose CL = 0.9, we find from the formula for s_{up}

$$s_{\text{up}} = -0.197 \quad (\text{CL} = 0.90)$$

Physicist:

We already knew $s \geq 0$ before we started; can't use negative upper limit to report result of expensive experiment!

Statistician:

The interval is designed to cover the true value only 90% of the time — this was clearly not one of those times.

Not uncommon dilemma when testing parameter values for which one has very little experimental sensitivity, e.g., very small s.

Expected limit for $s = 0$

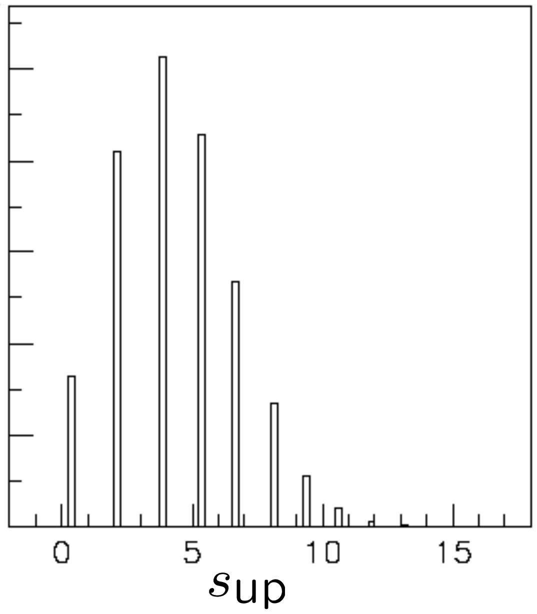
Physicist: I should have used CL = 0.95 — then $s_{up} = 0.496$

Even better: for CL = 0.917923 we get $s_{up} = 10^{-4}$!

Reality check: with $b = 2.5$, typical Poisson fluctuation in n is at least $\sqrt{2.5} = 1.6$. How can the limit be so low?

Look at the mean limit for the no-signal hypothesis ($s = 0$) (sensitivity).

Distribution of 95% CL limits with $b = 2.5$, $s = 0$.
Mean upper limit = 4.44



The Bayesian approach to limits

In Bayesian statistics need to start with 'prior pdf' $\pi(\theta)$, this reflects degree of belief about q before doing the experiment.

Bayes' theorem tells how our beliefs should be updated in light of the data x:

$$p(\theta|x) = \frac{L(x|\theta)\pi(\theta)}{\int L(x|\theta')\pi(\theta')\,d\theta'} \propto L(x|\theta)\pi(\theta)$$

Integrate posterior pdf $p(\theta|\,x)$ to give interval with any desired probability content.

For e.g. $n \sim$ Poisson($s+b$), 95% CL upper limit on s from

$$0.95 = \int_{-\infty}^{s_{\mathrm{up}}} p(s|n)\,ds$$

Bayesian prior for Poisson parameter

Include knowledge that $s \geq 0$ by setting prior $\pi(s) = 0$ for $s < 0$.

Could try to reflect 'prior ignorance' with e.g.

$$\pi(s) = \begin{cases} 1 & s \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Not normalized but this is OK as long as $L(s)$ dies off for large s.

Not invariant under change of parameter — if we had used instead a flat prior for, say, the mass of the Higgs boson, this would imply a non-flat prior for the expected number of Higgs events.

Doesn't really reflect a reasonable degree of belief, but often used as a point of reference;

or viewed as a recipe for producing an interval whose frequentist properties can be studied (coverage will depend on true s).

Bayesian upper limit with flat prior for *s*

Put Poisson likelihood and flat prior into Bayes' theorem:

$$p(s|n) \propto \frac{(s+b)^n}{n!} e^{-(s+b)} \qquad (s \geq 0)$$

Normalize to unit area:

$$p(s|n) = \frac{(s+b)^n e^{-(s+b)}}{\Gamma(b, n+1)}$$

upper incomplete gamma function

Upper limit s_{up} determined by requiring

$$1 - \alpha = \int_0^{s_{\mathrm{up}}} p(s|n)\, ds$$

Bayesian interval with flat prior for s

Solve to find limit s_{up}:

$$s_{\mathrm{up}} = \frac{1}{2} F_{\chi^2}^{-1}\left[p, 2(n+1)\right] - b$$

where

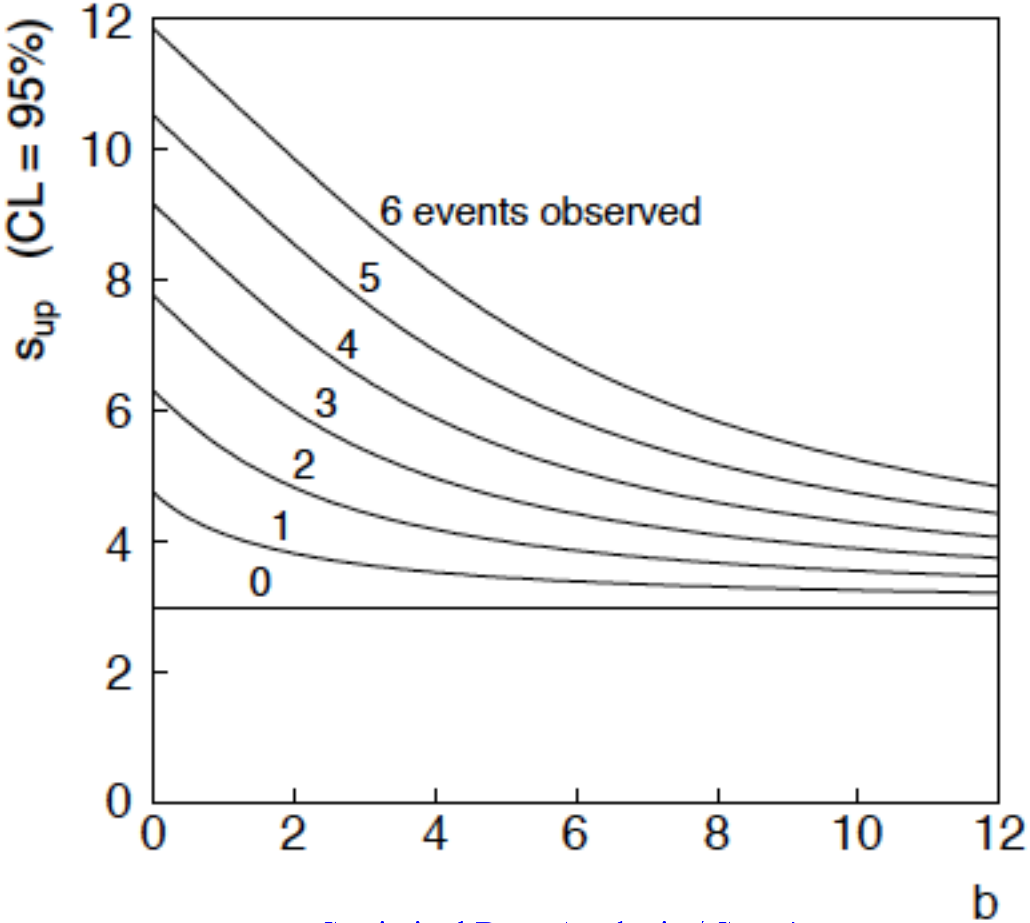
$$p = 1 - \alpha\left(1 - F_{\chi^2}\left[2b, 2(n+1)\right]\right)$$

For special case $b = 0$, Bayesian upper limit with flat prior numerically same as one-sided frequentist case ('coincidence').

Bayesian interval with flat prior for s

For $b > 0$ Bayesian limit is everywhere greater than the (one sided) frequentist upper limit.

Never goes negative. Doesn't depend on b if $n = 0$.



Priors from formal rules

Because of difficulties in encoding a vague degree of belief in a prior, one often attempts to derive the prior from formal rules, e.g., to satisfy certain invariance principles or to provide maximum information gain for a certain set of measurements.

Often called "objective priors"
Form basis of Objective Bayesian Statistics

The priors do not reflect a degree of belief (but might represent possible extreme cases).

In Objective Bayesian analysis, can use the intervals in a frequentist way, i.e., regard Bayes' theorem as a recipe to produce an interval with certain coverage properties.

Priors from formal rules (cont.)

For a review of priors obtained by formal rules see, e.g.,

Robert E. Kass and Larry Wasserman, *The Selection of Prior Distributions by Formal Rules*, J. Am. Stat. Assoc., Vol. 91, No. 435, pp. 1343-1370 (1996).

Formal priors have not been widely used in HEP, but there is recent interest in this direction, especially the reference priors of Bernardo and Berger; see e.g.

L. Demortier, S. Jain and H. Prosper, *Reference priors for high energy physics*, Phys. Rev. D 82 (2010) 034002, arXiv:1002.1111.

D. Casadei, *Reference analysis of the signal + background model in counting experiments*, JINST 7 (2012) 01012; arXiv:1108.4270.

Jeffreys' prior

According to *Jeffreys' rule*, take prior according to

$$\pi(\boldsymbol{\theta}) \propto \sqrt{\det(I(\boldsymbol{\theta}))}$$

where

$$I_{ij}(\boldsymbol{\theta}) = -E\left[\frac{\partial^2 \ln L(\boldsymbol{x}|\boldsymbol{\theta})}{\partial\theta_i \partial\theta_j}\right] = -\int \frac{\partial^2 \ln L(\boldsymbol{x}|\boldsymbol{\theta})}{\partial\theta_i \partial\theta_j} L(\boldsymbol{x}|\boldsymbol{\theta})\, d\boldsymbol{x}$$

is the Fisher information matrix.

One can show that this leads to inference that is invariant under a transformation of parameters.

For a Gaussian mean, the Jeffreys' prior is constant; for a Poisson mean μ it is proportional to $1/\sqrt{\mu}$.

Jeffreys' prior for Poisson mean

Suppose $n \sim$ Poisson(μ). To find the Jeffreys' prior for μ,

$$L(n|\mu) = \frac{\mu^n}{n!} e^{-\mu} \qquad \frac{\partial^2 \ln L}{\partial \mu^2} = -\frac{n}{\mu}$$

$$I = -E\left[\frac{\partial^2 \ln L}{\partial \mu^2}\right] = \frac{E[n]}{\mu^2} = \frac{1}{\mu}$$

$$\pi(\mu) \propto \sqrt{I(\mu)} = \frac{1}{\sqrt{\mu}}$$

So e.g. for $\mu = s + b$, this means the prior $\pi(s) \sim 1/\sqrt{(s + b)}$, which depends on b. But this is not designed as a degree of belief about s.

Prototype search analysis

Search for signal in a region of phase space; result is histogram of some variable *x* giving numbers:

$$\mathbf{n} = (n_1, \ldots, n_N)$$

Assume the n_i are Poisson distributed with expectation values

$$E[n_i] = \mu s_i + b_i$$

strength parameter

where

$$s_i = s_{\rm tot} \int_{{\rm bin}\, i} f_s(x; \boldsymbol{\theta}_s)\, dx \,, \qquad b_i = b_{\rm tot} \int_{{\rm bin}\, i} f_b(x; \boldsymbol{\theta}_b)\, dx \,.$$

signal

background

Prototype analysis (II)

Often also have a subsidiary measurement that constrains some of the background and/or shape parameters:

$$\mathbf{m} = (m_1, \ldots, m_M)$$

Assume the m_i are Poisson distributed with expectation values

$$E[m_i] = u_i(\boldsymbol{\theta})$$

nuisance parameters ($\boldsymbol{\theta}_s$, $\boldsymbol{\theta}_b$,b_{tot})

Likelihood function is

$$L(\mu, \boldsymbol{\theta}) = \prod_{j=1}^{N} \frac{(\mu s_j + b_j)^{n_j}}{n_j!} e^{-(\mu s_j + b_j)} \prod_{k=1}^{M} \frac{u_k^{m_k}}{m_k!} e^{-u_k}$$

Statistical tests for a non-established signal (i.e. a "search")

Consider a parameter μ proportional to the rate of a signal process whose existence is not yet established.

Suppose the model for the data includes both μ and a set of nuisance parameters $\boldsymbol{\theta}$.

To test hypothetical values of μ, use profile likelihood ratio:

$$\lambda(\mu) = \frac{L(\mu, \hat{\hat{\boldsymbol{\theta}}})}{L(\hat{\mu}, \hat{\boldsymbol{\theta}})}$$

maximizes L for specified μ

maximize L

Test statistic for discovery

Try to reject background-only ($\mu = 0$) hypothesis using

$$q_0 = \begin{cases} -2 \ln \lambda(0) & \hat{\mu} \geq 0 \\ 0 & \hat{\mu} < 0 \end{cases}$$

That is, here we regard positive m as the relevant alternative, so the critical region is taken to correspond to high values of $\hat{\mu}$.

Note that even though here physically $\mu \geq 0$, we allow to be negative. In large sample limit its distribution becomes Gaussian, and this will allow us to write down simple expressions for distributions of our test statistics.

Test statistic for upper limits

For purposes of setting an upper limit on μ use

$$q_\mu = \begin{cases} -2\ln\lambda(\mu) & \hat{\mu} \le \mu \\ 0 & \hat{\mu} > \mu \end{cases}$$

Here we regard the relevant alternative to be low μ, so we define the critical region to correspond to low values of $\hat{\mu}$.

Wald approximation for profile likelihood ratio

To find *p*-values, we need: $f(q_0|0), \quad f(q_\mu|\mu)$

For median significance under alternative, need: $f(q_\mu|\mu')$

Use approximation due to Wald (1943)

$$-2\ln\lambda(\mu) = \frac{(\mu-\hat{\mu})^2}{\sigma^2} + \mathcal{O}(1/\sqrt{N})$$

sample size

$$\hat{\mu} \sim \text{Gaussian}(\mu', \sigma)$$

$$\text{i.e., } E[\hat{\mu}] = \mu'$$

σ from covariance matrix V, use, e.g.,

$$V^{-1} = -E\left[\frac{\partial^2 \ln L}{\partial\theta_i \partial\theta_j}\right]$$

Noncentral chi-square for $-2\ln\lambda(\mu)$

If we can neglect the $O(1/\sqrt{N})$ term, $-2\ln\lambda(\mu)$ follows a noncentral chi-square distribution for one degree of freedom with noncentrality parameter

$$\Lambda = \frac{(\mu - \mu')^2}{\sigma^2}$$

As a special case, if $\mu' = \mu$ then $\Lambda = 0$ and $-2\ln\lambda(\mu)$ follows a chi-square distribution for one degree of freedom (Wilks).

Distribution of q_0 in large-sample limit

Assuming approximations valid in the large sample (asymptotic) limit, we can write down the full distribution of q_0 as

$$f(q_0|\mu') = \left(1 - \Phi\left(\frac{\mu'}{\sigma}\right)\right)\delta(q_0) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{q_0}}\exp\left[-\frac{1}{2}\left(\sqrt{q_0} - \frac{\mu'}{\sigma}\right)^2\right]$$

The special case $\mu' = 0$ is a "half chi-square" distribution:

$$f(q_0|0) = \frac{1}{2}\delta(q_0) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{q_0}}e^{-q_0/2}$$

In large sample limit, $f(q_0|0)$ independent of nuisance parameters; $f(q_0|\mu')$ depends on nuisance parameters through σ.

Cumulative distribution of q_0, significance

From the pdf, the cumulative distribution of q_0 is found to be

$$F(q_0|\mu') = \Phi\left(\sqrt{q_0} - \frac{\mu'}{\sigma}\right)$$

The special case $\mu' = 0$ is

$$F(q_0|0) = \Phi\left(\sqrt{q_0}\right)$$

The p-value of the $\mu = 0$ hypothesis is

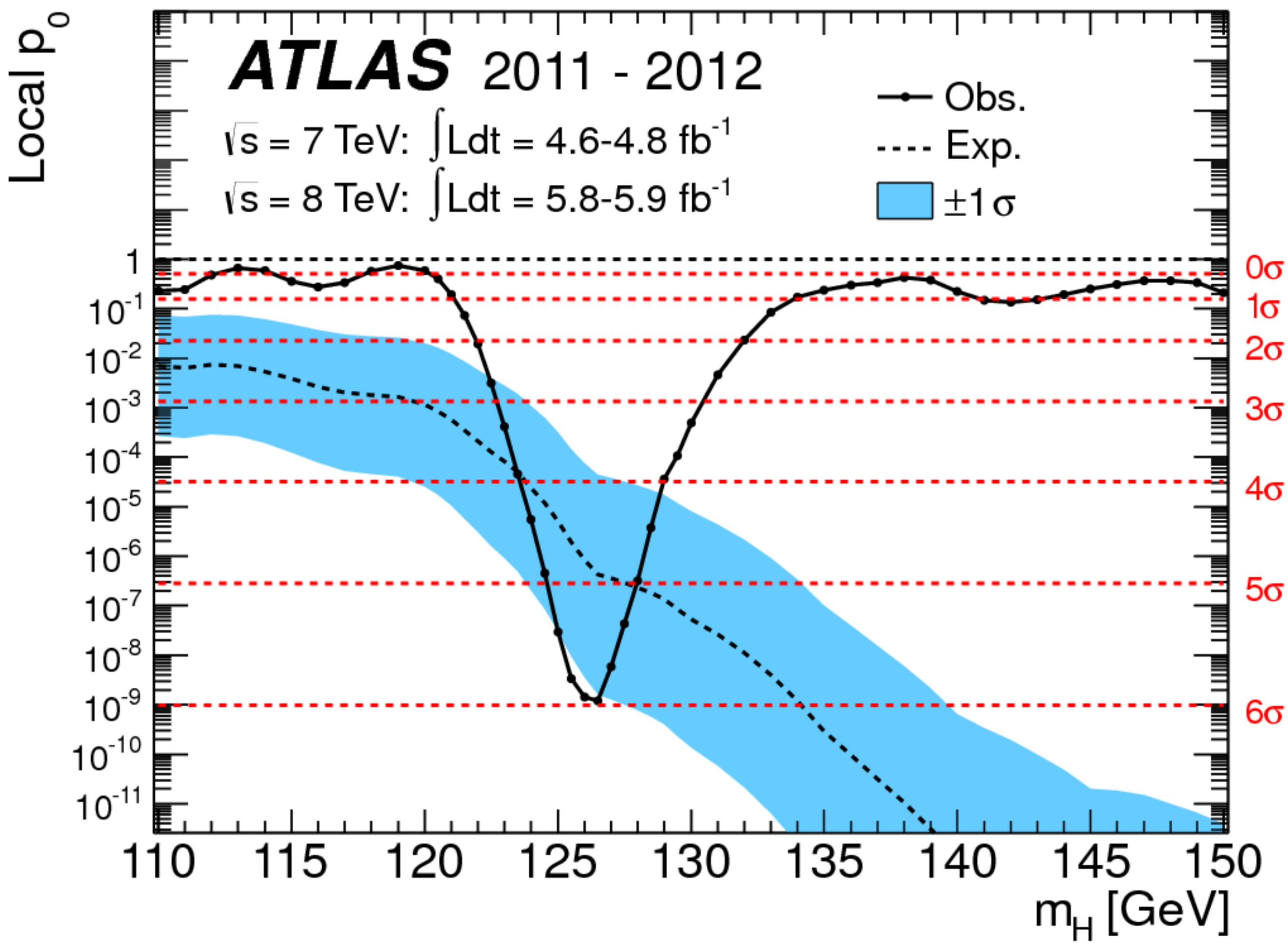
$$p_0 = 1 - F(q_0|0)$$

Therefore the discovery significance Z is simply

$$\boxed{Z = \Phi^{-1}(1 - p_0) = \sqrt{q_0}}$$

Example of a p-value

ATLAS, Phys. Lett. B 716 (2012) 1-29



Distribution of q_μ

Similar results for q_μ

$$f(q_\mu|\mu') = \Phi\left(\frac{\mu'-\mu}{\sigma}\right)\delta(q_\mu) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{q_\mu}}\exp\left[-\frac{1}{2}\left(\sqrt{q_\mu}-\frac{(\mu-\mu')}{\sigma}\right)^2\right]$$

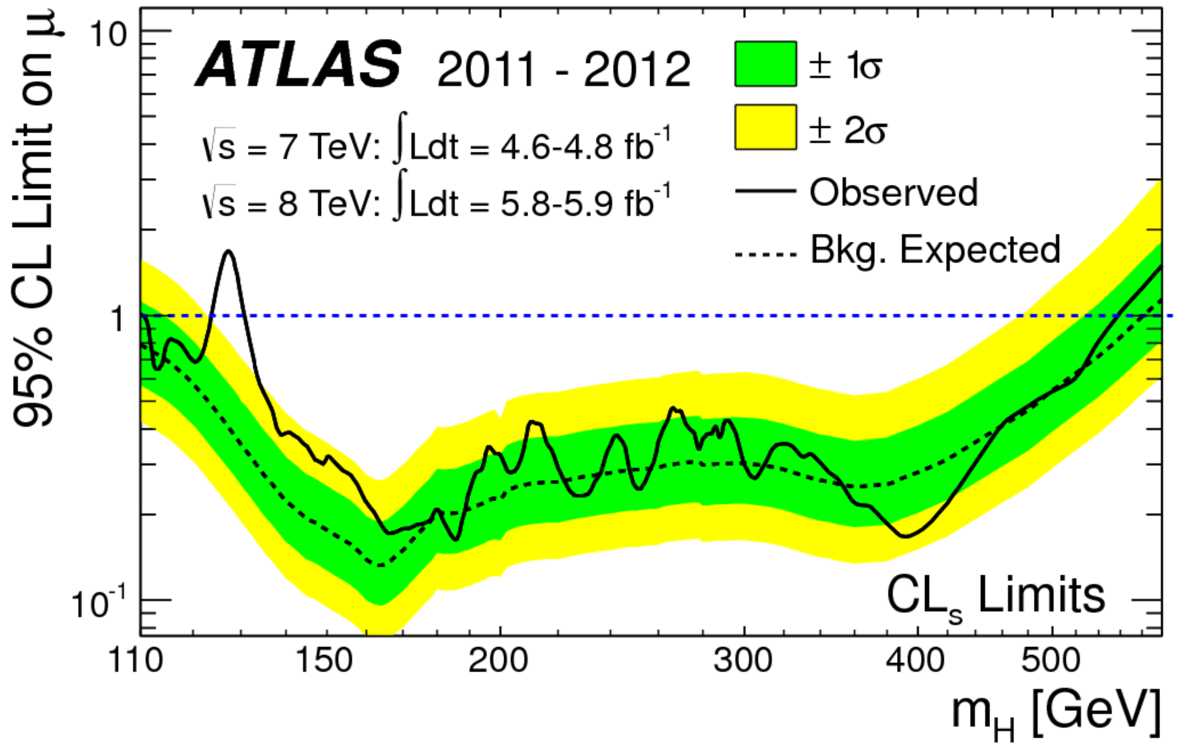
$$f(q_\mu|\mu) = \frac{1}{2}\delta(q_\mu) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{q_\mu}}e^{-q_\mu/2}$$

$$F(q_\mu|\mu') = \Phi\left(\sqrt{q_\mu}-\frac{(\mu-\mu')}{\sigma}\right)$$

$$p_\mu = 1 - F(q_\mu|\mu) = 1 - \Phi\left(\sqrt{q_\mu}\right)$$

Example of upper limits

ATLAS, Phys. Lett. B 716 (2012) 1-29



Not exactly what we described earlier as these are “CLs” limits; see, e.g., GDC, arXiv:1307.2487.

Profile likelihood with *b* uncertain

This is the well studied "on/off" problem: Cranmer 2005; Cousins, Linnemann, and Tucker 2008; Li and Ma 1983,...

Measure two Poisson distributed values:

$n \sim \text{Poisson}(s+b)$ (primary or "search" measurement)

$m \sim \text{Poisson}(\tau b)$ (control measurement, τ known)

The likelihood function is

$$L(s,b) = \frac{(s+b)^n}{n!} e^{-(s+b)} \frac{(\tau b)^m}{m!} e^{-\tau b}$$

Use this to construct profile likelihood ratio (b is nuisance parmeter):

$$\lambda(0) = \frac{L(0, \hat{\hat{b}}(0))}{L(\hat{s}, \hat{b})}$$

Ingredients for profile likelihood ratio

To construct profile likelihood ratio from this need estimators:

$$
\begin{aligned}
\hat{s} &= n - m/\tau \,, \\
\hat{b} &= m/\tau \,, \\
\hat{\hat{b}}(s) &= \frac{n + m - (1+\tau)s + \sqrt{(n + m - (1+\tau)s)^2 + 4(1+\tau)sm}}{2(1+\tau)} \,.
\end{aligned}
$$

and in particular to test for discovery ($s = 0$),

$$
\hat{\hat{b}}(0) = \frac{n+m}{1+\tau}
$$

Tests of asympotic formulae

Cowan, Cranmer, Gross, Vitells, EPJC 71 (2011) 1554; arXiv:1007.1727

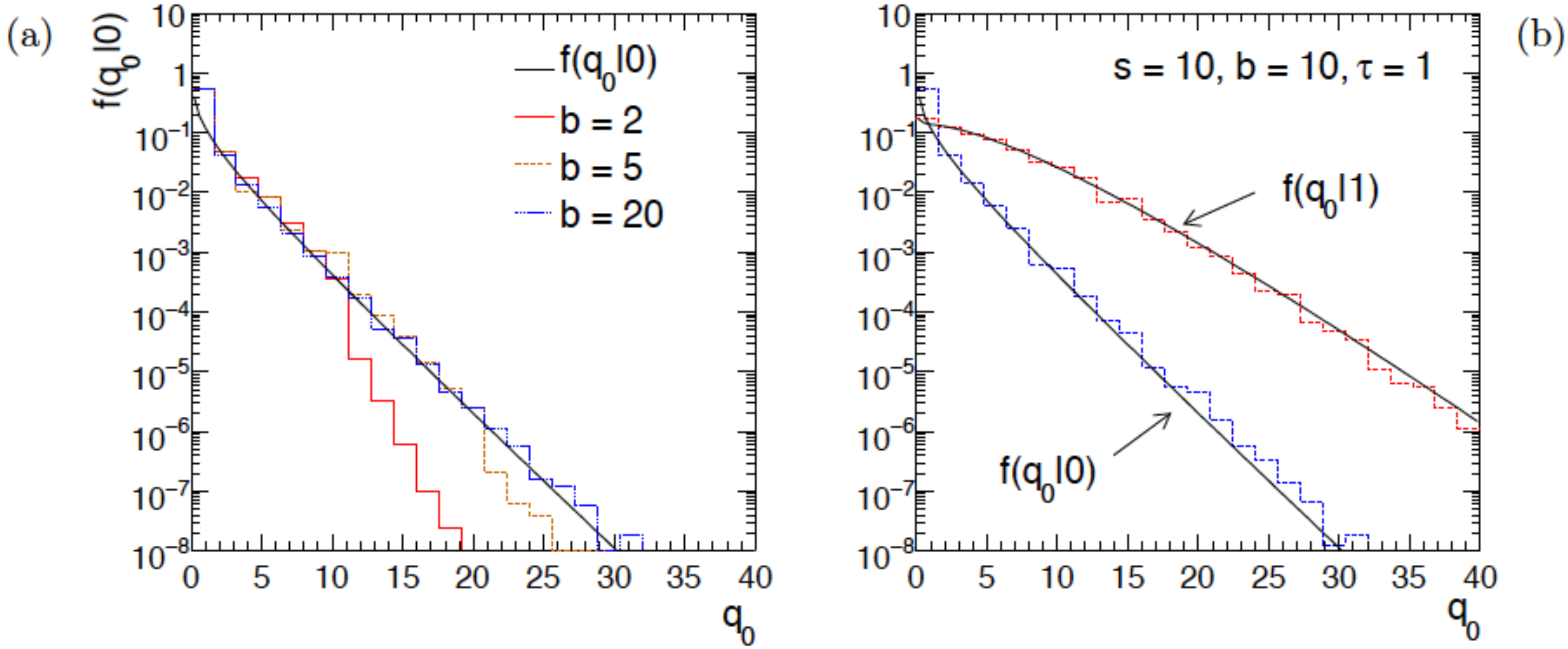


Figure 3: (a) The pdf $f(q_0|0)$ for the counting experiment. The solid curve shows $f(q_0|0)$ from Eq. (49) and the histograms are from Monte Carlo using different values of b (see text). (b) The distributions $f(q_0|0)$ and $f(q_0|1)$ from both the asymptotic formulae and Monte Carlo simulation based on $s = 10$, $b = 10$, $\tau = 1$.

Tests of asympotic formulae

Cowan, Cranmer, Gross, Vitells, EPJC 71 (2011) 1554; arXiv:1007.1727

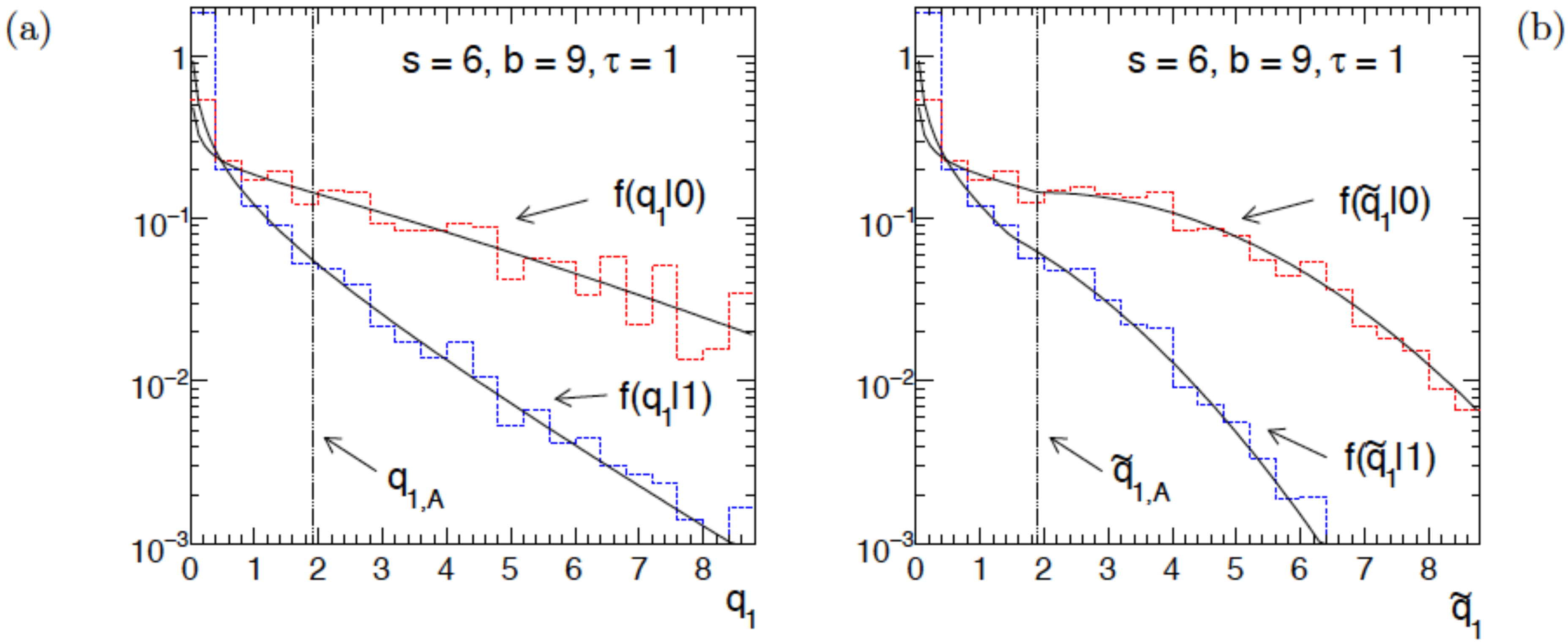
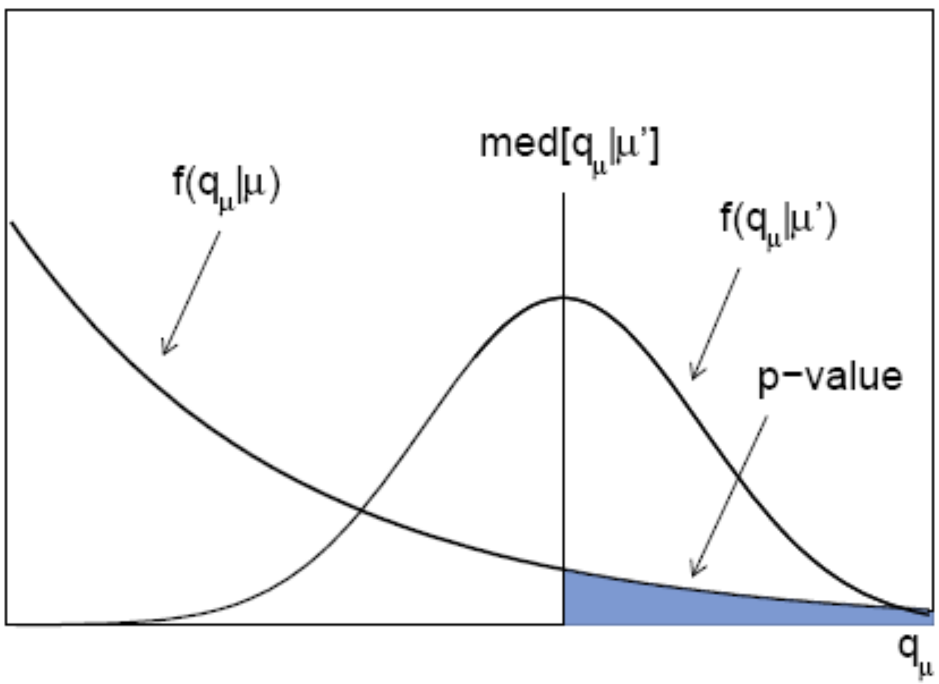


Figure 5: (a) The pdfs $f(q_1|1)$ and $f(q_1|0)$ for the counting experiment. The solid curves show the formulae from the text, and the histograms are from Monte Carlo using $s = 6$, $b = 9$, $\tau = 1$. (b) The same set of histograms with the alternative statistic $\tilde{q}_1$. The oscillatory structure evident in the histograms is a consequence of the discreteness of the data. The vertical line indicates the Asimov value of the test statistic corresponding to $\mu' = 0$.

Expected (or median) significance / sensitivity

When planning the experiment, we want to quantify how sensitive we are to a potential discovery, e.g., by given median significance assuming some nonzero strength parameter μ'.



So for p-value, need $f(q_0|0)$, for sensitivity, will need $f(q_0|\mu')$,

Expected discovery significance for counting experiment with background uncertainty

I. Discovery sensitivity for counting experiment with *b* known:

(a) $$\frac{s}{\sqrt{b}}$$

(b) Profile likelihood ratio test & Asimov: $$\sqrt{2\left((s+b)\ln\left(1+\frac{s}{b}\right)-s\right)}$$

II. Discovery sensitivity with uncertainty in *b*, σ_b:

(a) $$\frac{s}{\sqrt{b+\sigma_b^2}}$$

(b) Profile likelihood ratio test & Asimov:

$$\left[2\left((s+b)\ln\left[\frac{(s+b)(b+\sigma_b^2)}{b^2+(s+b)\sigma_b^2}\right]-\frac{b^2}{\sigma_b^2}\ln\left[1+\frac{\sigma_b^2 s}{b(b+\sigma_b^2)}\right]\right)\right]^{1/2}$$

Counting experiment with known background

Count a number of events $n \sim$ Poisson($s+b$), where

s = expected number of events from signal,

b = expected number of background events.

To test for discovery of signal compute p-value of $s = 0$ hypothesis,

$$p = P(n \geq n_{\rm obs}|b) = \sum_{n=n_{\rm obs}}^{\infty} \frac{b^n}{n!} e^{-b} = 1 - F_{\chi^2}(2b; 2n_{\rm obs})$$

Usually convert to equivalent significance: $Z = \Phi^{-1}(1-p)$
where Φ is the standard Gaussian cumulative distribution, e.g.,
$Z > 5$ (a 5 sigma effect) means $p < 2.9 \times 10^{-7}$.

To characterize sensitivity to discovery, give expected (mean or median) Z under assumption of a given s.

$s/\sqrt{b}$ for expected discovery significance

For large $s + b$, $n \to x \sim$ Gaussian(μ,σ) , $\mu = s + b$, $\sigma = \sqrt{(s + b)}$.

For observed value x_{obs}, p-value of $s = 0$ is Prob($x > x_{obs} \mid s = 0$),:

$$p_0 = 1 - \Phi\left(\frac{x_{\rm obs} - b}{\sqrt{b}}\right)$$

Significance for rejecting $s = 0$ is therefore

$$Z_0 = \Phi^{-1}(1 - p_0) = \frac{x_{\rm obs} - b}{\sqrt{b}}$$

Expected (median) significance assuming signal rate s is

$$\mathrm{median}[Z_0|s+b] = \frac{s}{\sqrt{b}}$$

Better approximation for significance

Poisson likelihood for parameter s is

$$L(s) = \frac{(s+b)^n}{n!} e^{-(s+b)}$$

For now no nuisance params.

To test for discovery use profile likelihood ratio:

$$q_0 = \begin{cases} -2 \ln \lambda(0) & \hat{s} \geq 0 \,, \\ 0 & \hat{s} < 0 \,. \end{cases}$$

$$\lambda(s) = \frac{L(s, \hat{\hat{\theta}}(s))}{L(\hat{s}, \hat{\theta})}$$

So the likelihood ratio statistic for testing $s = 0$ is

$$q_0 = -2 \ln \frac{L(0)}{L(\hat{s})} = 2 \left(n \ln \frac{n}{b} + b - n \right) \quad \text{for } n > b, \; 0 \text{ otherwise}$$

Approximate Poisson significance (continued)

For sufficiently large $s + b$, (use Wilks' theorem),

$$Z = \sqrt{2\left(n \ln \frac{n}{b} + b - n\right)} \quad \text{for } n > b \text{ and } Z = 0 \text{ otherwise.}$$

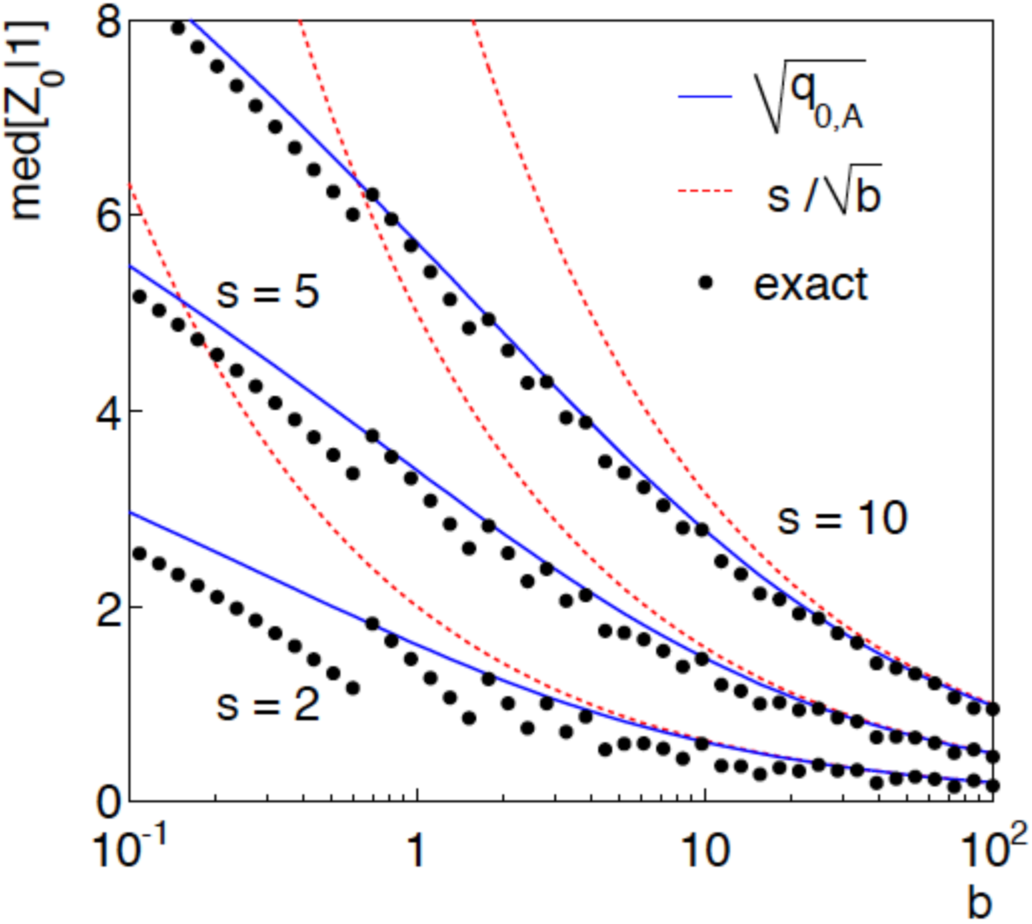
To find median[$Z|s$], let $n \rightarrow s + b$ (i.e., the Asimov data set):

$$Z_{\mathrm{A}} = \sqrt{2\left((s+b)\ln\left(1 + \frac{s}{b}\right) - s\right)}$$

This reduces to $s/\sqrt{b}$ for s << b.

$n \sim$ Poisson($s+b$), median significance, assuming s, of the hypothesis $s = 0$

CCGV, EPJC 71 (2011) 1554, arXiv:1007.1727



"Exact" values from MC, jumps due to discrete data.

Asimov $\sqrt{q_{0,A}}$ good approx. for broad range of s, b.

$s/\sqrt{b}$ only good for $s \ll b$.

Extending $s/\sqrt{b}$ to case where b uncertain

The intuitive explanation of $s/\sqrt{b}$ is that it compares the signal, s, to the standard deviation of n assuming no signal, $\sqrt{b}$.

Now suppose the value of b is uncertain, characterized by a standard deviation σ_b.

A reasonable guess is to replace $\sqrt{b}$ by the quadratic sum of $\sqrt{b}$ and σ_b, i.e.,

$$\mathrm{med}[Z|s] = \frac{s}{\sqrt{b + \sigma_b^2}}$$

This has been used to optimize some analyses e.g. where σ_b cannot be neglected.

Adding a control measurement for b

(The "on/off" problem: Cranmer 2005; Cousins, Linnemann, and Tucker 2008; Li and Ma 1983,...)

Measure two Poisson distributed values:

$n \sim \text{Poisson}(s+b)$ (primary or "search" measurement)

$m \sim \text{Poisson}(\tau b)$ (control measurement, τ known)

The likelihood function is

$$L(s,b) = \frac{(s+b)^n}{n!} e^{-(s+b)} \frac{(\tau b)^m}{m!} e^{-\tau b}$$

Use this to construct profile likelihood ratio (b is nuisance parmeter):

$$\lambda(0) = \frac{L(0, \hat{\hat{b}}(0))}{L(\hat{s}, \hat{b})}$$

Ingredients for profile likelihood ratio

To construct profile likelihood ratio from this need estimators:

$$\hat{s} = n - m/\tau ,$$

$$\hat{b} = m/\tau ,$$

$$\hat{\hat{b}}(s) = \frac{n + m - (1+\tau)s + \sqrt{(n + m - (1+\tau)s)^2 + 4(1+\tau)sm}}{2(1+\tau)} .$$

and in particular to test for discovery ($s = 0$),

$$\hat{\hat{b}}(0) = \frac{n+m}{1+\tau}$$

Asymptotic significance

Use profile likelihood ratio for q_0, and then from this get discovery significance using asymptotic approximation (Wilks' theorem):

$$
\begin{aligned}
Z &= \sqrt{q_0} \\
&= \left[-2 \left(n \ln \left[\frac{n+m}{(1+\tau)n} \right] + m \ln \left[\frac{\tau(n+m)}{(1+\tau)m} \right] \right) \right]^{1/2}
\end{aligned}
$$

for $n > \hat{b}$ and $Z = 0$ otherwise.

Essentially same as in:

Robert D. Cousins, James T. Linnemann and Jordan Tucker, NIM A 595 (2008) 480–501; arXiv:physics/0702156.

Tipei Li and Yuqian Ma, Astrophysical Journal 272 (1983) 317–324.

Asimov approximation for median significance

To get median discovery significance, replace n, m by their expectation values assuming background-plus-signal model:

$$n \rightarrow s + b$$

$$m \rightarrow \tau b$$

$$Z_{\mathrm{A}} = \left[-2 \left((s+b) \ln \left[\frac{s + (1+\tau)b}{(1+\tau)(s+b)} \right] + \tau b \ln \left[1 + \frac{s}{(1+\tau)b} \right] \right) \right]^{1/2}$$

Or use the variance of $\hat{b} = m/\tau$, $V[\hat{b}] \equiv \sigma_b^2 = \dfrac{b}{\tau}$, to eliminate τ:

$$\boxed{Z_{\mathrm{A}} = \left[2 \left((s+b) \ln \left[\frac{(s+b)(b+\sigma_b^2)}{b^2 + (s+b)\sigma_b^2} \right] - \frac{b^2}{\sigma_b^2} \ln \left[1 + \frac{\sigma_b^2 s}{b(b+\sigma_b^2)} \right] \right) \right]^{1/2}}$$

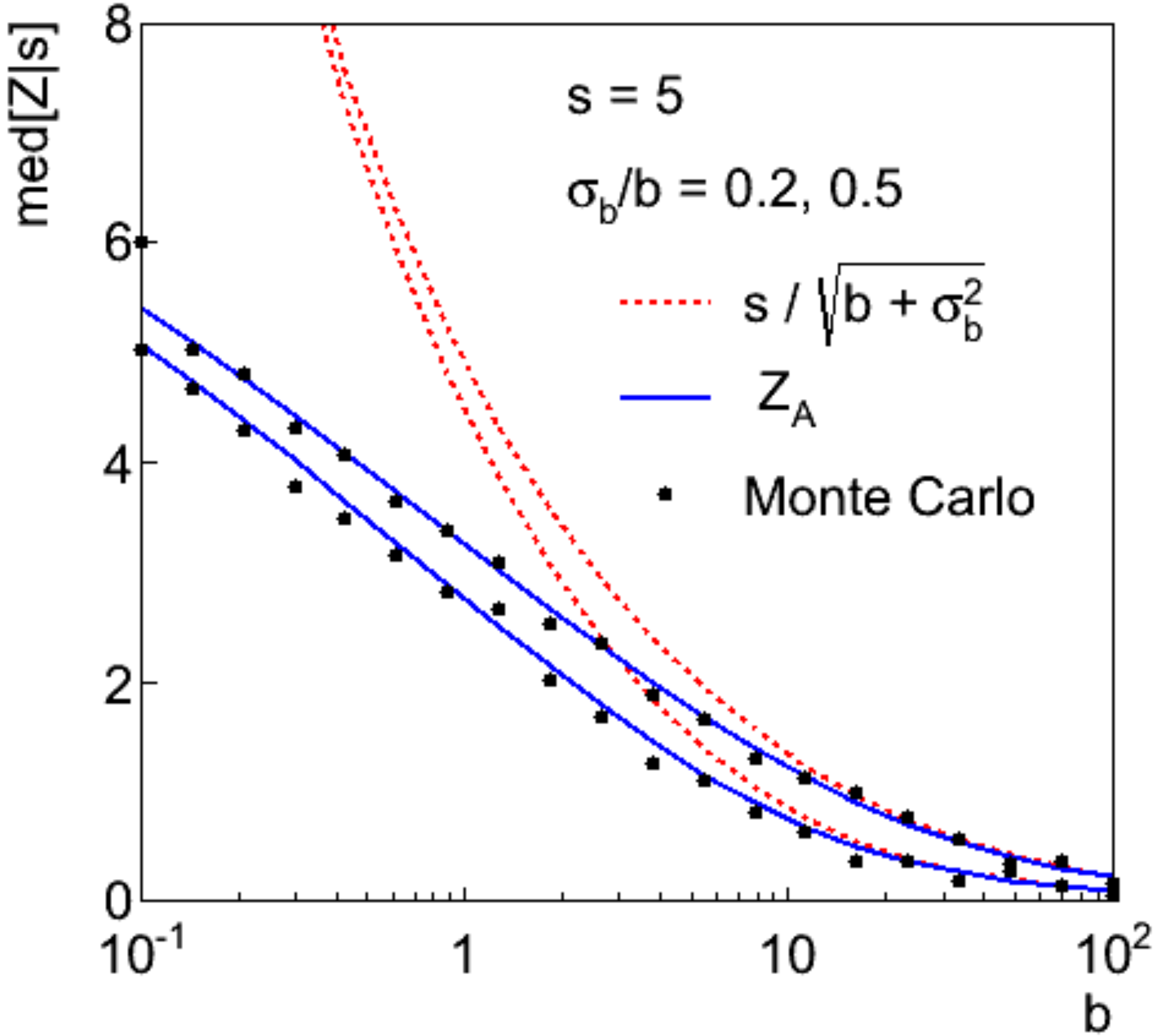
Limiting cases

Expanding the Asimov formula in powers of s/b and σ_b^2/b (= $1/\tau$) gives

$$Z_{\rm A} = \frac{s}{\sqrt{b + \sigma_b^2}} \left(1 + \mathcal{O}(s/b) + \mathcal{O}(\sigma_b^2/b)\right)$$

So the “intuitive” formula can be justified as a limiting case of the significance from the profile likelihood ratio test evaluated with the Asimov data set.

Testing the formulae: $s = 5$



Using sensitivity to optimize a cut

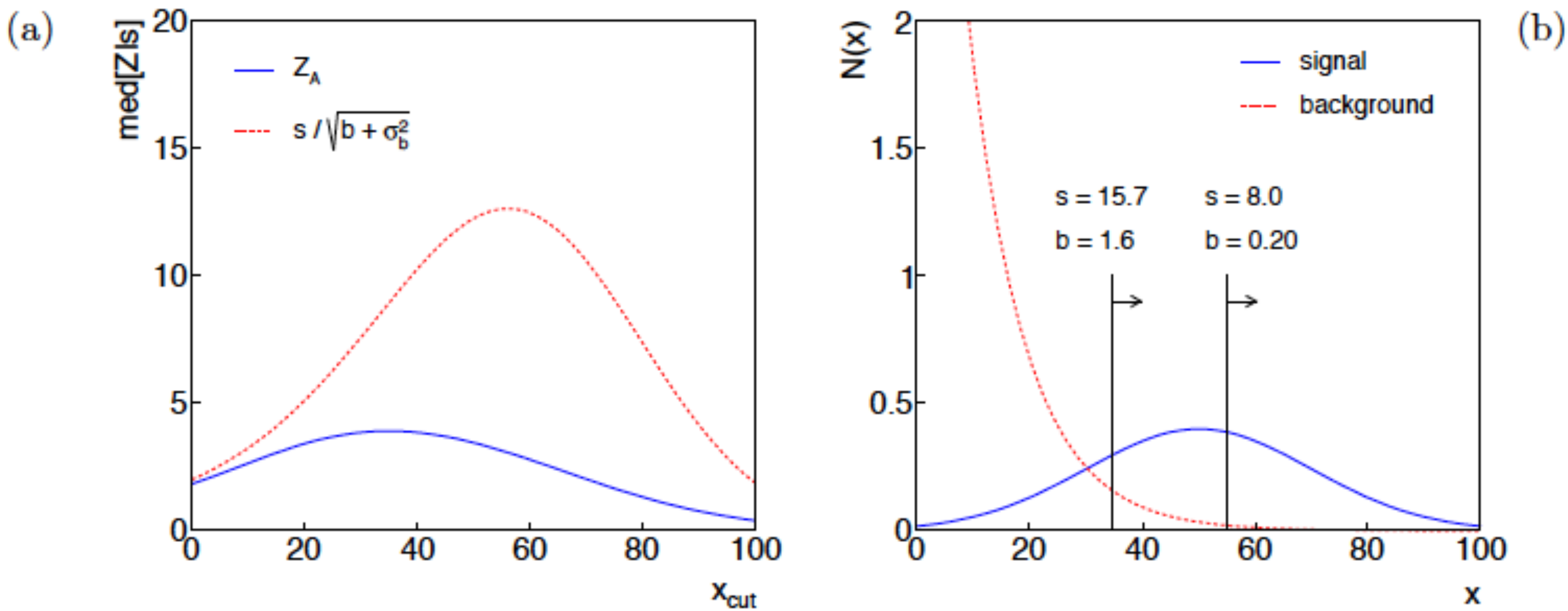


Figure 1: (a) The expected significance as a function of the cut value x_{cut}; (b) the distributions of signal and background with the optimal cut value indicated.

Return to interval estimation

Suppose a model contains a parameter μ; we want to know which values are consistent with the data and which are disfavoured.

Carry out a test of size α for all values of μ.

The values that are not rejected constitute a *confidence interval* for μ at confidence level CL = $1 - \alpha$.

The probability that the true value of μ will be rejected is not greater than α, so by construction the confidence interval will contain the true value of μ with probability $\geq 1 - \alpha$.

The interval depends on the choice of the test (critical region).

If the test is formulated in terms of a p-value, p_μ, then the confidence interval represents those values of μ for which $p_\mu > \alpha$.

To find the end points of the interval, set $p_\mu = \alpha$ and solve for μ.

Test statistic for upper limits

cf. Cowan, Cranmer, Gross, Vitells, arXiv:1007.1727, EPJC 71 (2011) 1554.

For purposes of setting an upper limit on μ one can use

$$q_\mu = \begin{cases} -2\ln\lambda(\mu) & \hat{\mu} \le \mu \\ 0 & \hat{\mu} > \mu \end{cases} \qquad \text{where} \qquad \lambda(\mu) = \frac{L(\mu, \hat{\hat{\boldsymbol{\theta}}})}{L(\hat{\mu}, \hat{\boldsymbol{\theta}})}$$

I.e. when setting an upper limit, an upwards fluctuation of the data is not taken to mean incompatibility with the hypothesized μ:

From observed q_μ find p-value: $\quad p_\mu = \int_{q_{\mu,\mathrm{obs}}}^{\infty} f(q_\mu|\mu)\, dq_\mu$

Large sample approximation: $\quad \boxed{p_\mu = 1 - \Phi\left(\sqrt{q_\mu}\right)}$

95% CL upper limit on μ is highest value for which p-value is not less than 0.05.

Monte Carlo test of asymptotic formulae

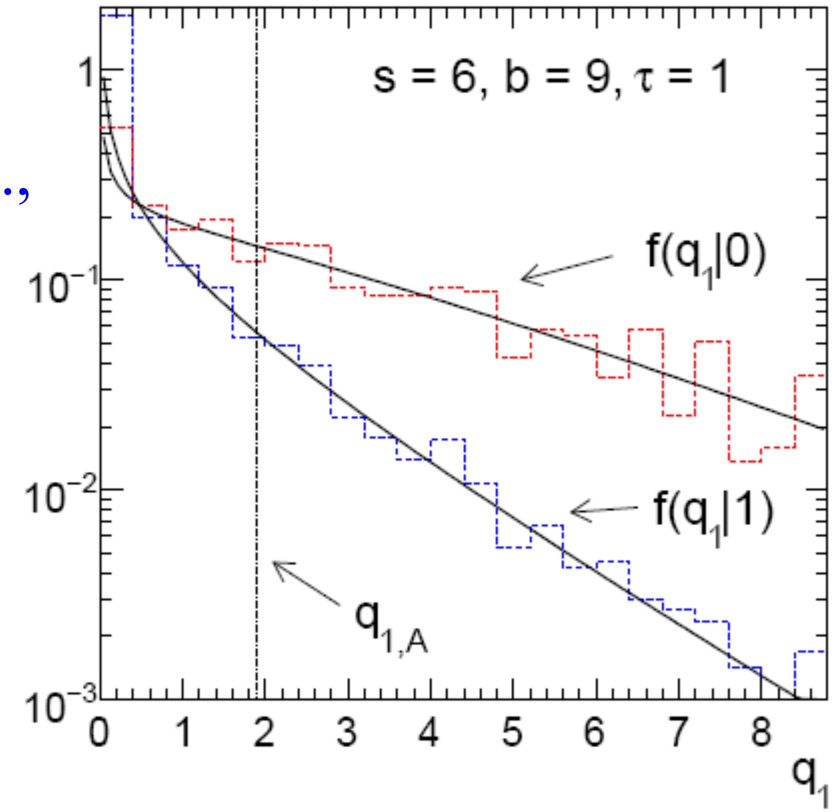
Consider again $n \sim$ Poisson $(\mu s + b)$, $m \sim$ Poisson(τb)
Use q_μ to find p-value of hypothesized μ values.

E.g. $f(q_1|1)$ for p-value of μ=1.

Typically interested in 95% CL, i.e., p-value threshold = 0.05, i.e., $q_1 = 2.69$ or $Z_1 = \sqrt{q_1} = 1.64$.

Median[q_1 |0] gives "exclusion sensitivity".

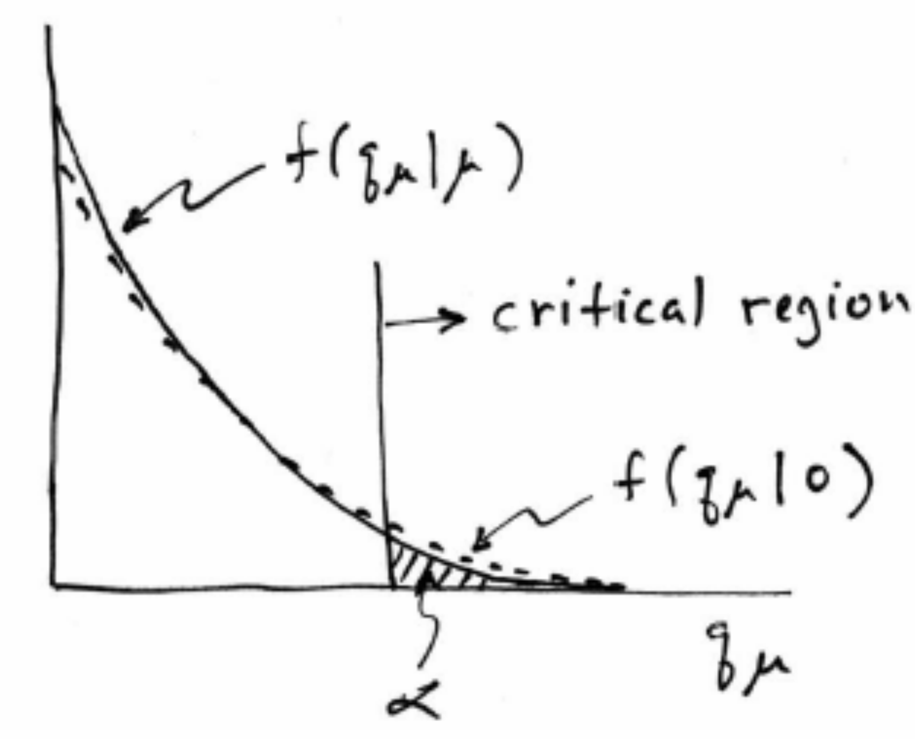
Here asymptotic formulae good for $s = 6$, $b = 9$.



Low sensitivity to μ

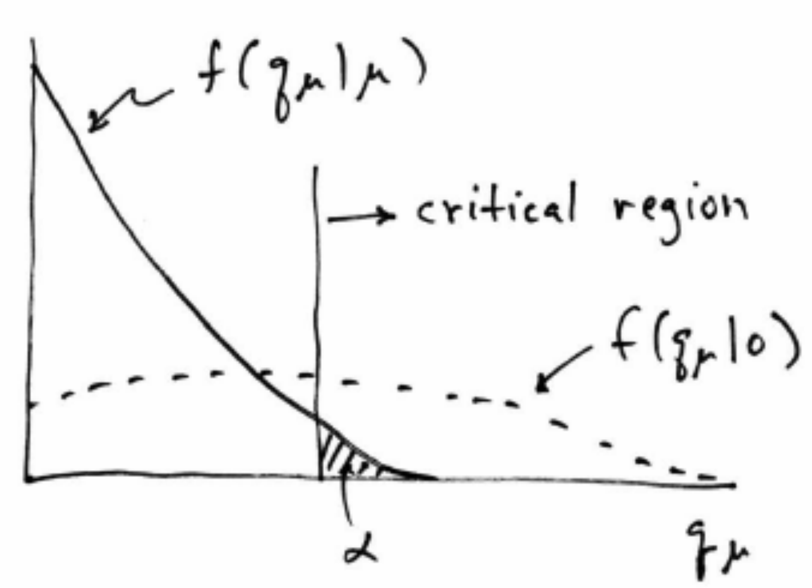
It can be that the effect of a given hypothesized μ is very small relative to the background-only ($\mu = 0$) prediction.

This means that the distributions $f(q_\mu|\mu)$ and $f(q_\mu|0)$ will be almost the same:



Having sufficient sensitivity

In contrast, having sensitivity to μ means that the distributions $f(q_\mu|\mu)$ and $f(q_\mu|0)$ are more separated:

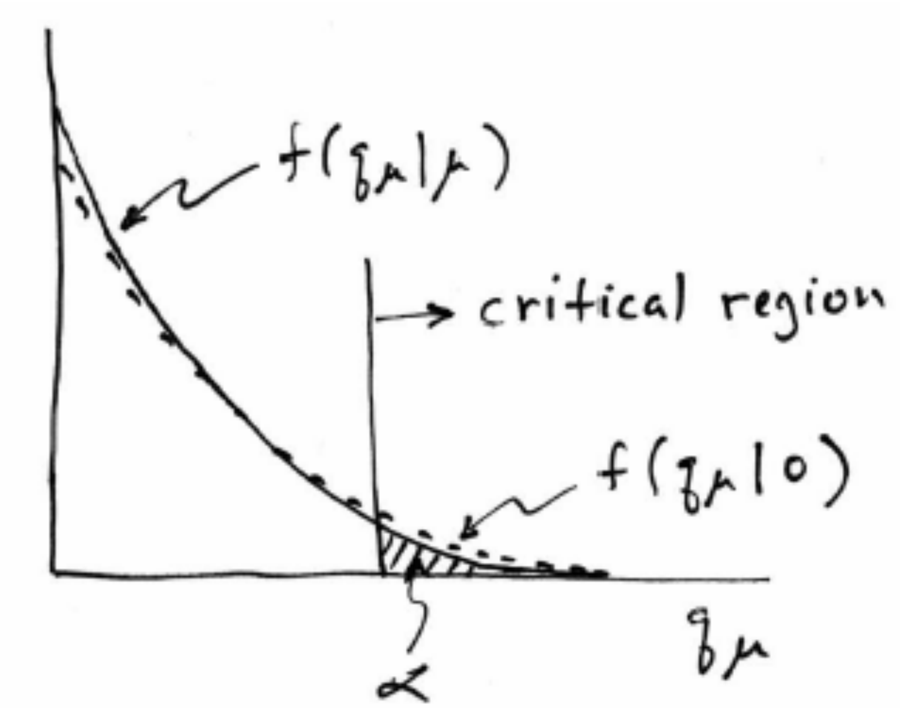


That is, the power (probability to reject μ if $\mu = 0$) is substantially higher than α. Use this power as a measure of the sensitivity.

Spurious exclusion

Consider again the case of low sensitivity. By construction the probability to reject μ if μ is true is α (e.g., 5%).

And the probability to reject μ if $\mu = 0$ (the power) is only slightly greater than α.



This means that with probability of around $\alpha = 5\%$ (slightly higher), one excludes hypotheses to which one has essentially no sensitivity (e.g., $m_{\mathrm{H}} = 1000$ TeV).

"Spurious exclusion"

Ways of addressing spurious exclusion

The problem of excluding parameter values to which one has no sensitivity known for a long time; see e.g.,

Virgil L. Highland, *Estimation of Upper Limits from Experimental Data*, July 1986, Revised February 1987, Temple University Report C00-3539-38.

In the 1990s this was re-examined for the LEP Higgs search by Alex Read and others

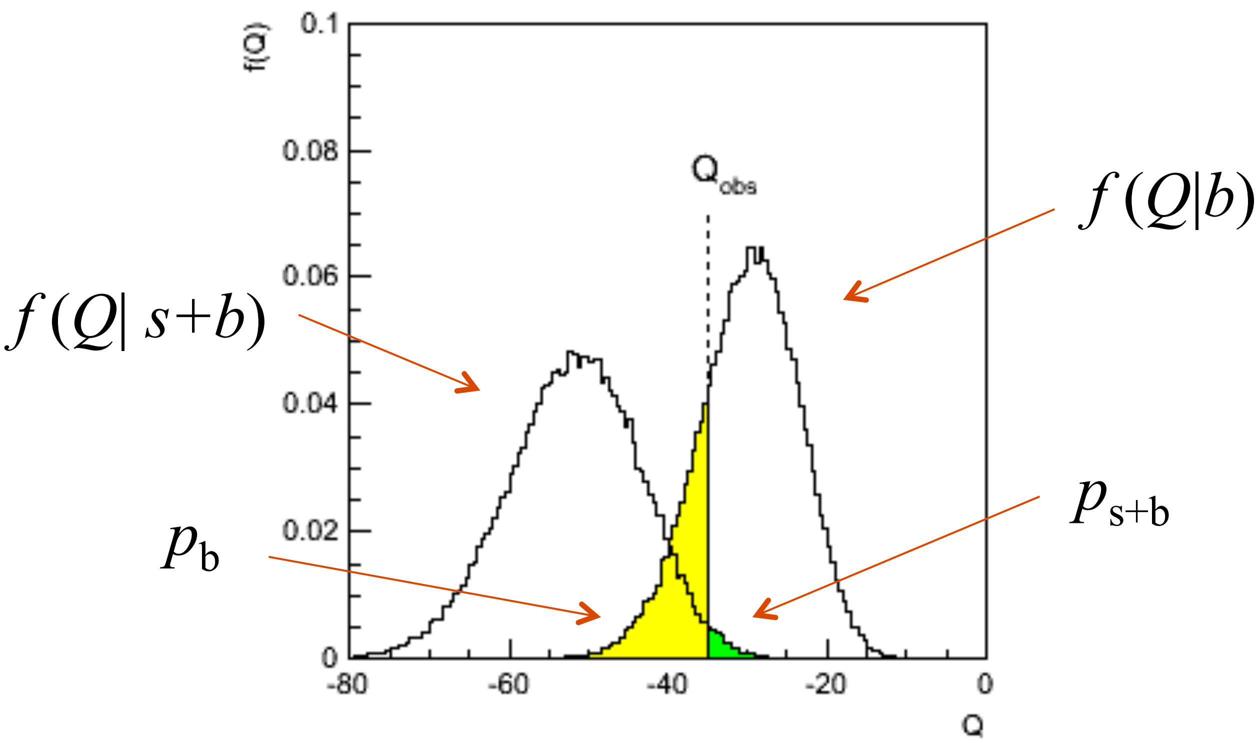
T. Junk, Nucl. Instrum. Methods Phys. Res., Sec. A **434**, 435 (1999); A.L. Read, J. Phys. G **28**, 2693 (2002).

and led to the "CL$_s$" procedure for upper limits.

Unified intervals also effectively reduce spurious exclusion by the particular choice of critical region.

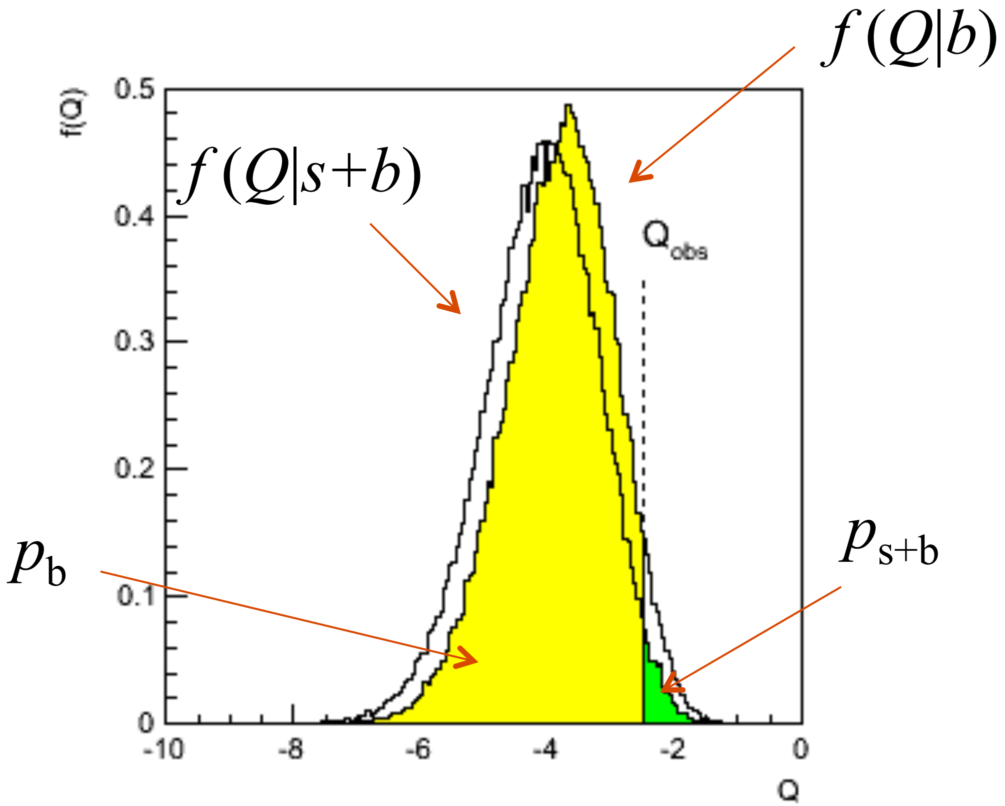
The CL$_s$ procedure

In the usual formulation of CL$_s$, one tests both the $\mu = 0$ (b) and $\mu > 0$ ($\mu s+b$) hypotheses with the same statistic $Q = -2\ln L_{s+b}/L_b$:



The CL$_s$ procedure (2)

As before, "low sensitivity" means the distributions of Q under b and s+b are very close:



The CL$_s$ procedure (3)

The CL$_s$ solution (A. Read et al.) is to base the test not on the usual p-value (CL$_{s+b}$), but rather to divide this by CL$_b$ (~ one minus the p-value of the b-only hypothesis), i.e.,

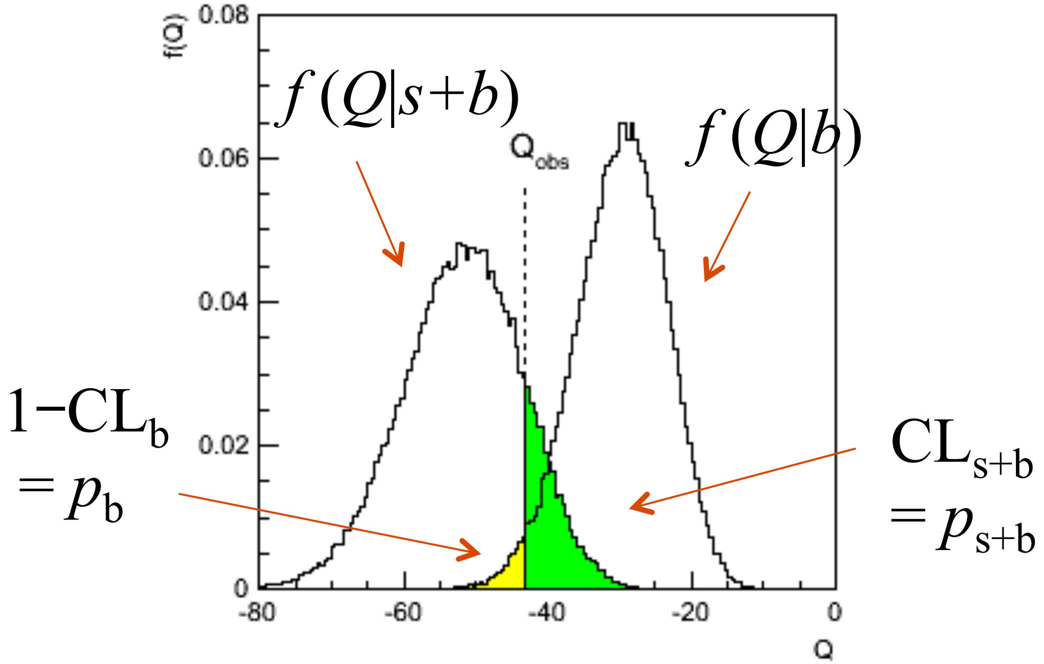
Define:

$$\mathrm{CL_s} = \frac{\mathrm{CL_{s+b}}}{\mathrm{CL_b}}$$

$$= \frac{p_{s+b}}{1 - p_b}$$

Reject s+b hypothesis if:

$$\mathrm{CL_s} \le \alpha$$



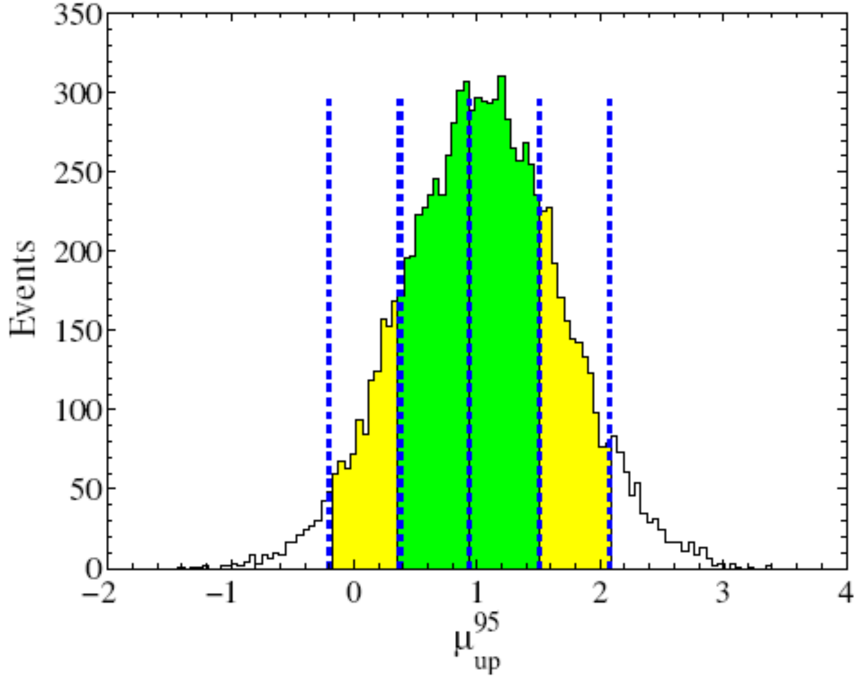
Increases "effective" p-value when the two distributions become close (prevents exclusion if sensitivity is low).

Setting upper limits on $\mu = \sigma/\sigma_{SM}$

Carry out the CLs procedure for the parameter $\mu = \sigma/\sigma_{SM}$, resulting in an upper limit μ_{up}.

In, e.g., a Higgs search, this is done for each value of m_H.

At a given value of m_H, we have an observed value of μ_{up}, and we can also find the distribution $f(\mu_{up}|0)$:



$\pm 1\sigma$ (green) and $\pm 2\sigma$ (yellow) bands from toy MC;

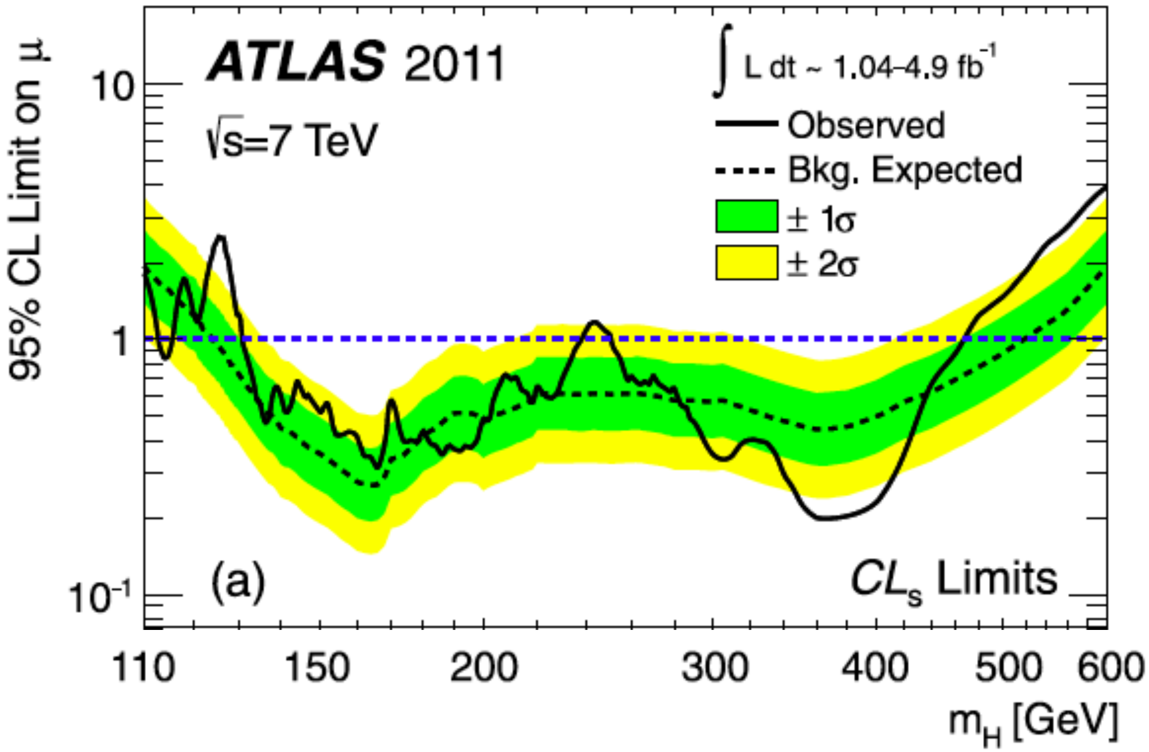
Vertical lines from asymptotic formulae.

How to read the green and yellow limit plots

For every value of m_{H}, find the CLs upper limit on μ.

Also for each m_{H}, determine the distribution of upper limits μ_{up} one would obtain under the hypothesis of $\mu = 0$.

The dashed curve is the median μ_{up}, and the green (yellow) bands give the $\pm$ 1σ (2σ) regions of this distribution.



ATLAS, Phys. Lett. B 710 (2012) 49-66

Choice of test for limits (2)

In some cases $\mu = 0$ is no longer a relevant alternative and we want to try to exclude μ on the grounds that some other measure of incompatibility between it and the data exceeds some threshold.

If the measure of incompatibility is taken to be the likelihood ratio with respect to a two-sided alternative, then the critical region can contain both high and low data values.

→ unified intervals, G. Feldman, R. Cousins,
Phys. Rev. D 57, 3873–3889 (1998)

The Big Debate is whether to use one-sided or unified intervals in cases where small (or zero) values of the parameter are relevant alternatives. Professional statisticians have voiced support on both sides of the debate.

Unified (Feldman-Cousins) intervals

We can use directly

$$t_\mu = -2 \ln \lambda(\mu) \qquad \text{where} \qquad \lambda(\mu) = \frac{L(\mu, \hat{\hat{\boldsymbol{\theta}}})}{L(\hat{\mu}, \hat{\boldsymbol{\theta}})}$$

as a test statistic for a hypothesized μ.

Large discrepancy between data and hypothesis can correspond either to the estimate for μ being observed high or low relative to μ.

This is essentially the statistic used for Feldman-Cousins intervals (here also treats nuisance parameters).

G. Feldman and R.D. Cousins, Phys. Rev. D 57 (1998) 3873.

Lower edge of interval can be at $\mu = 0$, depending on data.

Distribution of t_μ

Using Wald approximation, $f(t_\mu|\mu')$ is noncentral chi-square for one degree of freedom:

$$f(t_\mu|\mu') = \frac{1}{2\sqrt{t_\mu}} \frac{1}{\sqrt{2\pi}} \left[\exp\left(-\frac{1}{2} \left(\sqrt{t_\mu} + \frac{\mu - \mu'}{\sigma} \right)^2 \right) + \exp\left(-\frac{1}{2} \left(\sqrt{t_\mu} - \frac{\mu - \mu'}{\sigma} \right)^2 \right) \right]$$

Special case of $\mu = \mu'$ is chi-square for one d.o.f. (Wilks).

The p-value for an observed value of t_μ is

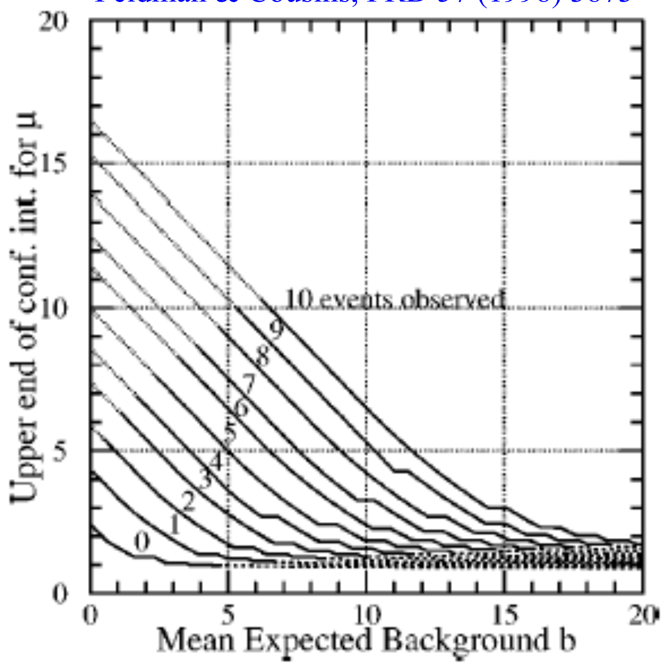
$$p_\mu = 1 - F(t_\mu|\mu) = 2\left(1 - \Phi\left(\sqrt{t_\mu}\right)\right)$$

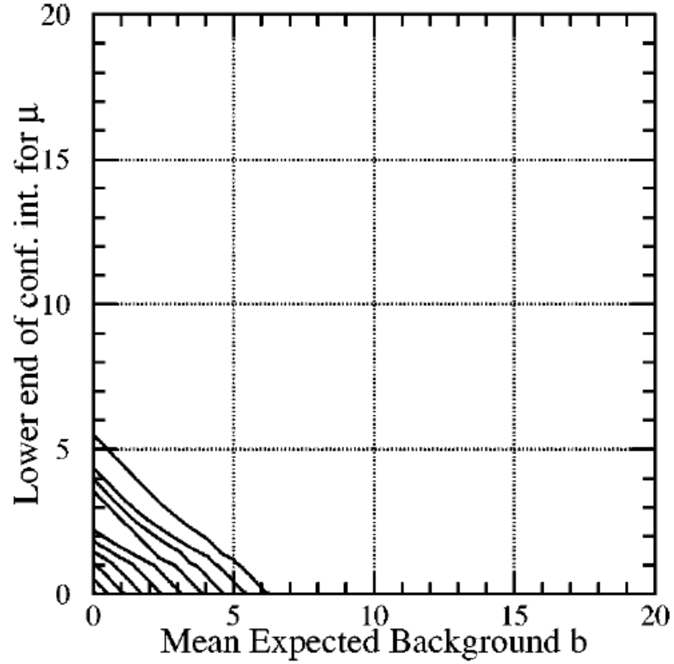
and the corresponding significance is

$$Z_\mu = \Phi^{-1}(1 - p_\mu) = \Phi^{-1}\left(2\Phi\left(\sqrt{t_\mu}\right) - 1\right)$$

Upper/lower edges of F-C interval for μ versus b for $n \sim$ Poisson($\mu+b$)

Feldman & Cousins, PRD 57 (1998) 3873





Lower edge may be at zero, depending on data.

For $n = 0$, upper edge has (weak) dependence on b.

Feldman-Cousins discussion

The initial motivation for Feldman-Cousins (unified) confidence intervals was to eliminate null intervals.

The F-C limits are based on a likelihood ratio for a test of μ with respect to the alternative consisting of all other allowed values of μ (not just, say, lower values).

The interval's upper edge is higher than the limit from the one-sided test, and lower values of μ may be excluded as well. A substantial downward fluctuation in the data gives a low (but nonzero) limit.

This means that when a value of μ is excluded, it is because there is a probability α for the data to fluctuate either high or low in a manner corresponding to less compatibility as measured by the likelihood ratio.