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Statistical Analysis of Data
1998/99 postgraduate exam solutions

(a) The likelihood function is the product of the Poisson probabilities,

$$L(\theta) = \prod_{i=1}^{N} \frac{\nu_i^{n_i}}{n_i!} e^{-\nu_i} ,$$

where $\nu_i = \theta x_i$. Taking the logarithm and dropping the terms that do not depend on θ gives the log-likelihood function,

$$\log L(\theta) = \sum_{i=1}^{N} (n_i \log \nu_i - \nu_i) .$$

Setting the derivative with respect to θ equal to zero to find the maximum gives

$$\hat{\theta} = \frac{\sum_{i=1}^N n_i}{\sum_{i=1}^N x_i} .$$

(b) To compute the bias of $\hat{\theta}$, use the fact that for Poisson variables, the expectation value is equal to the mean, i.e. $E[n_i] = \nu_i = \theta x_i$. This gives

$$E[\hat{\theta}] = E\left[\frac{\sum_{i=1}^N n_i}{\sum_{i=1}^N x_i}\right] = \frac{\sum_{i=1}^N E[n_i]}{\sum_{i=1}^N x_i} = \frac{\sum_{i=1}^N \theta x_i}{\sum_{i=1}^N x_i} = \theta$$

and thus $b = E[\hat{\theta}] - \theta = 0$.

(c) To compute the variance of $\hat{\theta}$, use the fact that for Poisson variables the variance is equal to the mean, i.e. $V[n_i] = \nu_i = \theta x_i$. Recall also that the variance of a sum of independent random variables is equal to the sum of variances, and that when we pull a constant outside the variance it gets squared. Applying these rules to $\hat{\theta}$ gives

$$V[\hat{\theta}] = V\left[\frac{\sum_{i=1}^N n_i}{\sum_{i=1}^N x_i}\right] = \frac{\sum_{i=1}^N V[n_i]}{\left(\sum_{i=1}^N x_i\right)^2} = \frac{\sum_{i=1}^N \theta x_i}{\left(\sum_{i=1}^N x_i\right)^2} = \frac{\theta}{\sum_{i=1}^N x_i} .$$

To obtain the RCF bound, we need the expectation value of the second derivative of $\log L$. This is

$$E\left[\frac{\partial^2 \log L}{\partial \theta^2}\right] = E\left[\sum_{i=1}^N \frac{-n_i}{\theta^2}\right] = \frac{-\sum_{i=1}^N \nu_i}{\theta^2} = \frac{-\sum_{i=1}^N x_i}{\theta} .$$

The RCF bound is thus given by

$$\text{RCF bound} = \frac{-1}{E\left[\frac{\partial^2 \log L}{\partial \theta^2}\right]} = \frac{\theta}{\sum_{i=1}^N x_i} \; ,$$

which is the same as the variance $V[\hat{\theta}]$ we have computed above.

(d) To find the LS estimator, first construct the quantity

$$\chi^2(\theta) = \sum_{i=1}^{N} \frac{(n_i - \nu_i)^2}{\nu_i} = \sum_{i=1}^{N} \frac{(n_i - \theta x_i)^2}{\theta x_i} \; ,$$

where in the denominators we have used $\sigma_i^2 = \nu_i$ for the variances of Poisson variables. Setting the derivative of $\chi^2(\theta)$ equal to zero to find the minimum,

$$\frac{\partial \chi^2}{\partial \theta} = \sum_{i=1}^{N} \left(\frac{n_i^2}{\theta^2 x_i} - x_i \right) = 0 \; ,$$

gives the LS estimator

$$\hat{\theta} = \left[\frac{\sum_{i=1}^N (n_i^2 / x_i)}{\sum_{i=1}^N x_i} \right]^{1/2} \; .$$

(e) To evaluate the goodness-of-fit we need to define a statistic t whose value reflects the level of agreement between the data and the hypothesis. From this we can compute the P-value, i.e. the probability to obtain t with an equal or worse level of agreement between data and hypothesis than the observed value t_{obs}.

For the statistic t in conjunction with LS, the χ^2 as defined in (d) can be used. With ML we can use the following statistic discussed in the lectures,

$$\chi^2_{\mathrm{P}} = 2 \sum_{i=1}^{N} \left(n_i \log \frac{n_i}{\hat{\nu}_i} + \hat{\nu}_i - n_i \right) \; ,$$

where $\hat{\nu}_i = \hat{\theta} x_i$. Both of these will follow a χ^2 distribution in the large sample limit, i.e. all $\nu_i \gg 1$. If this is not the case, the pdf can be determined by Monte Carlo; this can then be used to find the P-value, i.e. the probability $P(\chi^2 \geq \chi^2_{\mathrm{obs}})$.

(f) Applying the variance from (c) to the case here gives

$$V[\hat{\theta}] = \frac{\theta}{E_1(L - L_2) + E_2 L_2} \; .$$

If $E_1 > E_2$, this is minimum for $L_2 = 0$; if $E_1 < E_2$, use $L_2 = L$. That is, for purposes of minimizing the variance of $\hat{\theta}$, all of the data should be taken at the highest energy. This can be seen intuitively by realizing that θ is the slope of a line that passes through zero, and the best determination of the slope is made using the longest 'lever arm'.