PH2150A Scientific Computing Skills Team Project
Optical Aberrations of Spherical and Parabolic Mirrors

1 Introduction

In this project you will investigate optical aberrations that occur in reflecting telescopes. You will do this by simulating the transport of rays of light (or photons) through a telescope and determining how well they focus to a point. First you will specify for the rays an angle of incidence relative to the optical axis of the telescope. Then you simulate the trajectories of a large number of photons that enter at this angle but at different locations over the surface of the telescope's mirror. This is what you would obtain from a distant point source of light, such as a star, that is not in the centre of the telescope's field of view. In an ideal telescope, these rays would all be focussed at a single point. For the practical optical systems that you will consider, however, the point is blurred. For each simulated photon you can determine where it crosses the focal plane, and in this way you can investigate the nature of the blurring and explore how to minimize it.

2 The assignment

Your assignment is to investigate the blurring of a distant point source of light in reflecting telescopes with spherical and parabolic mirrors. You will need to write a program that simulates the production of light rays at a specified incident angle but random location over the surface of the mirror, trace these rays through the optical system and see where they cross the focal plane. You will need to:

- specify the geometry of the mirror and the incident angle of the light rays;
- use an algorithm based on random numbers to generate the location of an incident photon;
- determine where the photon strikes the mirror;
- determine the trajectory of the reflected photon;
- calculate where the reflected photon traverses the focal plane of the mirror;
- repeat this procedure for many photons at the same angle but different incident locations;
- quantify the size of the aberrations for different optical systems as a function of the incident angle.

3 Spherical aberration, coma, astigmatism

If we consider parallel rays of light incident on a spherical mirror with a large radius of curvature R, then these will all arrive at approximately the same *focal point*, a distance $F = R/2$ from the mirror. If the radius of curvature is not very large compared to the mirror's size, however, then the rays will not be focussed at a point, as indicated schematically in Fig. 1(a); this is called *spherical aberration*. Spherical aberration can be corrected by making the mirror parabolic, as indicated in Fig. 1(b).

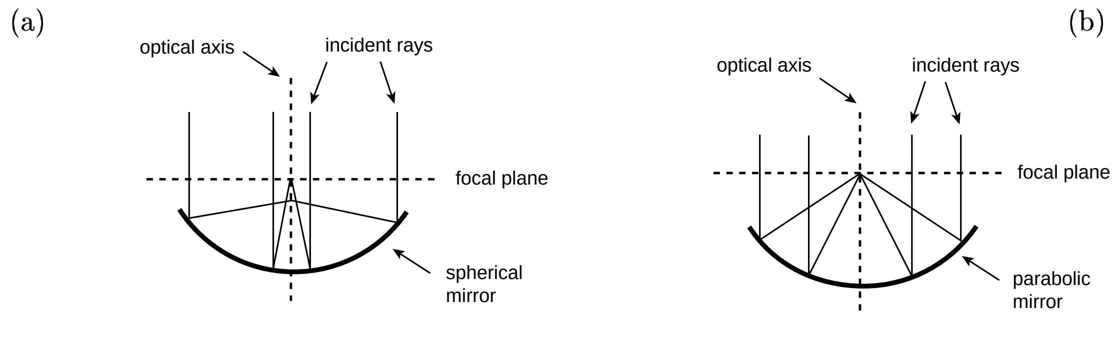

Figure 1: Schematic illustration of (a) spherical aberration and (b) its correction by use of a parabolic mirror.

The problem with parabolic mirrors is that they suffer from a second type of aberration called *coma*. This is where incoming rays that are not parallel to the optical axis are not focused at a point, as indicated in Fig. 2(a). The image from a point source of light is smeared out into blob as shown in Fig. 2(b). In the figure, the dots represent rays of light all parallel to each other, entering the telescope with a non-zero angle relative to the optical axis. Most large telescopes, such as the 200-inch Palomar reflector, have parabolic mirrors and for them the coma-free field of view is very narrow. They are well-suited for objects of very small angular size such as individual stars or distant galaxies. They cannot, however, be used to survey large portions of the sky or to view extended objects such as large nearby galaxies. The Andromeda Galaxy as viewed from Earth, for example, subtends several degrees.

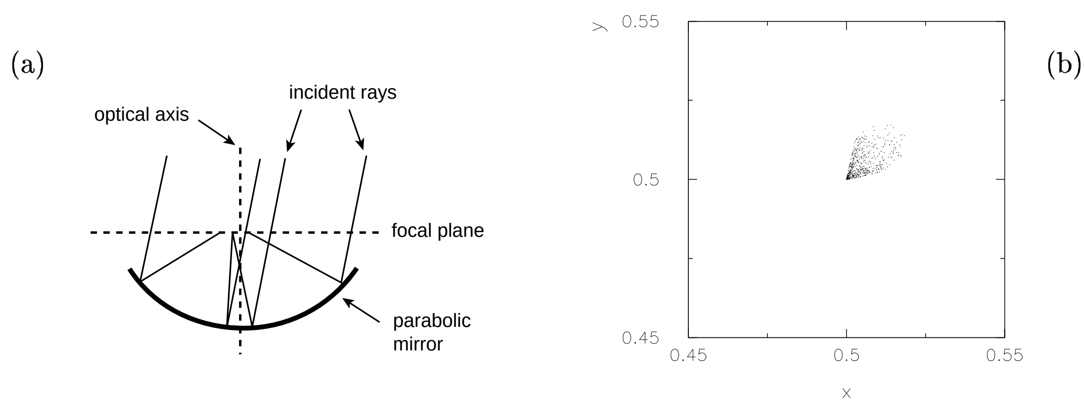

Figure 2: (a) Schematic illustration of rays from a distant point source, i.e., parallel to each other, but not parallel to the optical axis. (b) The image formed by a parabolic mirror showing the distortion due to coma.

A type of telescope invented by Bernhard Schmidt in 1932 solves the problem of coma by using a spherical mirror. Here it doesn't matter what direction the rays enter relative to the mirror's axis, since a spherical surface is symmetric about any axis that passes through the centre of the sphere. The spherical aberration is cured by introducing a thin lens called a Schmidt corrector plate. The *Schmidt–Cassegrain* telescope in the RHUL observatory is of this type. As an extension objective you should consider how to modify your program to determine the optimal shape of a Schmidt corrector plate.

Finally, it can happen that a parabolic mirror is not perfectly symmetric about the optical axis but rather is described by, say, a parabola of a certain focal length along one direction, but of a slightly different focal length along the perpendicular direction. This is called *astigmatism*, and occurs in optical telescopes as well as in human eyes.

4 Ray tracing

A surface in space can be described by the set of points such that $f(x, y, z) = c$ for some function f and constant c. For example, a spherical mirror with its base at $z = 0$ pointing upwards along the z axis is given by

$$f_{\mathrm{s}}(x, y, z) \equiv z + \sqrt{4F^2 - x^2 - y^2} = 2F \,, \tag{1}$$

where $F = R/2$ is the focal length. A parabolic mirror is described by

$$f_{\mathrm{p}}(x, y, z) \equiv z - \frac{x^2 + y^2}{4F} = 0 \,. \tag{2}$$

For a parabola with astigmatism, this becomes

$$f_{\mathrm{a}}(x, y, z) = z - \frac{x^2}{4F_x} - \frac{y^2}{4F_y} = 0 \,. \tag{3}$$

A light ray can be described by a point in space through which the ray passes and a direction. The set of points on the ray can be expressed by a vector of the form

$$\mathbf{V}(t) = \mathbf{V}_0 + \mathbf{v}t \,, \tag{4}$$

where $\mathbf{V}_0$ specifies a location, $\mathbf{v}$ is a unit vector giving the ray's direction, and t is a parameter whose value determines the position on the ray.

To generate a ray of light you begin by finding a random position in the (x, y) plane between certain limits, e.g.,

$$x_0 = (x_{\mathrm{max}} - x_{\mathrm{min}})r_1 + x_{\mathrm{min}} \,, \tag{5}$$

$$y_0 = (y_{\mathrm{max}} - y_{\mathrm{min}})r_2 + y_{\mathrm{min}} \,, \tag{6}$$

where x_{min}, x_{max}, y_{min} and y_{max} are specified limits that are large enough to cover the mirror and r_1 and r_2 are independent random numbers each uniformly distributed between zero and one. The incident ray can then be described by

$$\mathbf{I}(t) = \mathbf{I}_0 + \mathbf{i}t \,, \tag{7}$$

where $\mathbf{I}_0 = (x_0, y_0, z_0)$ with x_0 and y_0 as determined above and z_0 can be taken as zero, as indicated in Fig. 3. Note that this is not the actual place where the light is produced, and indeed the photon never arrives at (x_0, y_0, z_0) as it is below the mirror. This doesn't matter, however, since all we need to define the ray is a point anywhere on the line. For practical reasons it is convenient to take z_0 to be zero.

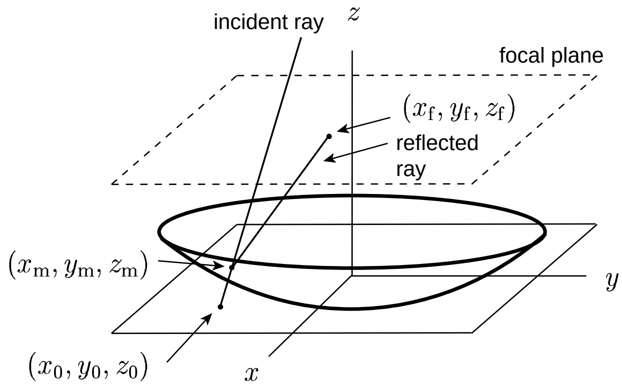

Figure 3: Schematic illustration of ray tracing (see text).

You must specify the unit vector $\mathbf{i}$ for the incident rays. For example, $\mathbf{i} = (0, 0, 1)$ describes a vertical ray; $(0, 0.01, 0.9999)$ has an angle of approximately 0.01 radians relative to the z axis. Note that it doesn't matter whether we take the unit vector to point up or down; its sign is not important.

To find where a ray will hit the mirror, you need to determine the value of the parameter t such that

$$f(I_x(t), I_y(t), I_z(t)) = c \,, \tag{8}$$

where the relation $f(x, y, z) = c$ describes the surface of the mirror, e.g., equations (1), (2) or (3). Once t has been found this is substituted into equation (7) to give the point $\mathbf{R}_{\mathrm{m}} = (x_{\mathrm{m}}, y_{\mathrm{m}}, z_{\mathrm{m}})$ where the ray intersects the surface of the mirror. Let us suppose that the mirror has a radius of r_{m}, which of course is smaller (typically much smaller) than the radius of curvature of its surface. If you find $\sqrt{x_{\mathrm{m}}^2 + y_{\mathrm{m}}^2} < r_{\mathrm{m}}$, then the photon did indeed hit the mirror; if not, it missed, in which case you reject the photon and try again.

The vector normal to a surface that is described by $f(x, y, z) = c$ for any constant c is proportional to ∇f. For the parabolic mirror, for example, this is

$$\mathbf{n} \propto \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \left(\frac{-x}{2F}, \frac{-y}{2F}, 1 \right) \,. \tag{9}$$

In order for this to be a unit vector, we need to divide by its norm, i.e.,

$$\mathbf{n} \rightarrow \mathbf{n} / \sqrt{n_x^2 + n_y^2 + n_z^2} \,. \tag{10}$$

Once we have the normalized vector $\mathbf{n}$, we can determine the unit vector $\mathbf{r}$ for the reflected ray. Both $\mathbf{i}$ and $\mathbf{r}$ can be expressed as a sum of two vectors parallel and perpendicular to the normal vector $\mathbf{n}$, i.e., $\mathbf{i} = \mathbf{i}_{||} + \mathbf{i}_{\perp}$ and $\mathbf{r} = \mathbf{r}_{||} + \mathbf{r}_{\perp}$, as shown in Fig. 4.

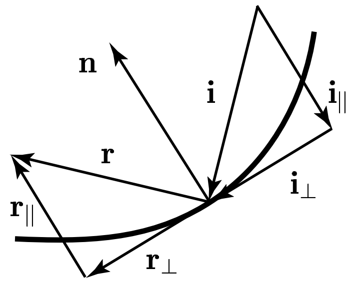

Figure 4: The relationship between the unit vectors **i**, **n** and **r** (see text).

From the law of reflection we have $\mathbf{r}_\perp = \mathbf{i}_\perp$ and $\mathbf{r}_{||} = -\mathbf{i}_{||}$, as can be seen from the figure. The magnitude of $\mathbf{i}_{||}$ is $\mathbf{i} \cdot \mathbf{n}$ and its direction is parallel to $\mathbf{n}$, so it is therefore given by $\mathbf{i}_{||} = (\mathbf{i} \cdot \mathbf{n})\mathbf{n}$. Putting these ingredients together gives

$$\begin{aligned} \mathbf{r} &= -\mathbf{i}_{||} + \mathbf{i}_\perp \\ &= -\mathbf{i}_{||} + (\mathbf{i} - \mathbf{i}_{||}) \\ &= \mathbf{i} - 2(\mathbf{i} \cdot \mathbf{n})\mathbf{n} \,. \end{aligned} \tag{11}$$

Once $\mathbf{r}$ has been found it is straightforward to determine the value of the parameter t at which $\mathbf{R}(t) = \mathbf{R}_\mathrm{m} + \mathbf{r}t$ intersects the focal plane at $z = F$, and thus to obtain the corresponding coordinates $(x_\mathrm{f}, y_\mathrm{f}, z_\mathrm{f})$.

For a given incident direction **i** this can be repeated for a large number of photons generated at random locations over the surface of the mirror. As a measure of the focusing ability of the telescope, you can compute the RMS (root mean square) spot size s. This is given by

$$s = \left[\frac{1}{n} \sum_{i=1}^{n} \left((x_i - \overline{x})^2 + (y_i - \overline{y})^2 \right) \right]^{1/2} , \tag{12}$$

where $\overline{x}$ and $\overline{y}$ are the average values of x_f and y_f for the generated set of rays.

If you have enough time, you may wish to extend your program to include lenses, in which case you will have to model the effect of refraction at an interface between, say, air and glass. In a manner similar to above for reflection, one can shown that the unit vector **r** describing a refracted ray is given by

$$\mathbf{r} = \frac{n_\mathrm{i}}{n_\mathrm{r}} \mathbf{i} - \left(\frac{n_\mathrm{i}}{n_\mathrm{r}} (\mathbf{i} \cdot \mathbf{n}) + \sqrt{1 - \left(\frac{n_\mathrm{i}}{n_\mathrm{r}} \right)^2 (1 - (\mathbf{i} \cdot \mathbf{n})^2)} \right) \mathbf{n} \,, \tag{13}$$

where n_i and n_r are the indices of refraction for the media of the incident and refracted rays and the normal vector **n** is defined to point into the medium of the incident ray.

5 Teamwork

You will need to divide up the work so that each member has specific tasks to accomplish and the team achieves its goals to the fullest extent possible. Below are some suggestions on possible ways of separating the project into individual tasks.

In Java you will probably define a class called, say, `Ray`, which defines a ray of light. This will contain data members for the unit vector that specifies its direction, as well as a point on the ray. The `Ray` object can then be passed to a function that computes where it intersects with a spherical or parabolic mirror and thus determines the normal vector at this point. Another function can then find the reflected ray according to equation (11).

1. In the *main* method, set up the problem by specifying the telescope's geometry and an incident angle for the rays relative to the optical axis.
2. Create an object of type `Ray`.
3. Implement a method that finds where the ray intersects with the mirror. From this determine the vector normal to the mirror at this point.
4. Write a method that computes the reflected ray.
5. Write a method that determines where the reflected ray crosses the focal plane.

You should write the main method that creates `Ray` objects and executes the various methods in the desired way. You may wish to define a `Mirror` class with a method to calculate the normal unit vector given a point on its surface. For different mirrors you could either implement different methods of the same class or you could define different classes with methods of the same name. Alternatively you can create a class that contains all of the various utility methods.

Once the position where the ray crosses the focal point has been determined, this should be recorded, e.g., in an array, so that it can be passed to a plotting routine. The values can be used to compute the RMS spot size s as a function of the incident angle.

If time permits you should create plotting routines that display individual rays. You should consider a number of different diameters and focal lengths; as examples you can find on the web the dimensions of real telescopes such as the 200-inch Palomar reflector.

6 References

Optical telescopes are described in:

C.R. Kitchin, *Astrophysical Techniques*, 3rd edition, IoP Publishing (1998).

There are many resources on ray tracing on the internet, especially in the context of computer graphics. A readable paper is:

Paul Rademacher, *Ray Tracing: Graphics for the Masses*, (1997)
`www.cs.unc.edu/~rademach/xroads-RT/RTarticle.html` .

G. Cowan, 19 November 2001