PH2150A Scientific Computing Skills Team Project
Solving the Laplace equation using the relaxation method

1 Introduction

In this project you will solve the Laplace equation,

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \, , \tag{1}$$

using a numerical method call *relaxation* (also called the Jacobi method). This is an iterative technique which involves finding successive approximations to the solution on a grid of points. This type of problem comes up, for example, when finding the electrical potential in a volume with given boundary conditions.

Specifically, consider a volume of square cross section in the x, y plane ($0 \leq x \leq a$ and $0 \leq y \leq a$) and which is infinitely long in the z direction. One side ($y = a$) is held at a potential φ_0, the the other three sides are held at $\varphi = 0$, and inside the volume is a vacuum. The analytic solution to the problem, even for this relatively simple geometry, is not trivial to derive. It is

$$\varphi(x, y) = \sum_{n \,\mathrm{odd}} A_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \, , \tag{2}$$

where

$$A_n = \frac{4\varphi_0}{n\pi \sinh(n\pi)} \, . \tag{3}$$

Analytic solutions can become much more difficult (or impossible) to find for more complicated boundary conditions. So it is very useful to know how to solve problems like this numerically.

2 The assignment

Your assignment is to investigate solving the Laplace equation using the relaxation method, and specifically to find the potential everywhere inside the volume described above. Note that by symmetry the potential does not depend on z, and so we are really dealing with a two-dimensional problem. The tasks include the following:

- find the potential $\varphi(x, y)$ with the relaxation method;
- given $\varphi(x, y)$, find the equipotential lines and produce plots;
- investigate the rate of convergence and numerical errors of the solution (this should include a comparison with the exact solution);
- investigate the dependence of the numerical solution on the grid spacing.

If this is successfully completed, you can try more complicated geometries and boundary conditions.

3 The relaxation method

In the relaxation method, one calculates the potential at points on a grid in the x, y plane, as shown in Fig. 1. The idea is to convert the Laplace equation into an equation relating the value of φ at each point on the grid to the value at neighbouring points. We will then start with a guess for φ everywhere on the grid, (even a bad guess, as long as it satisfies the boundary conditions), and we will update the estimated solution at each point using the values of neighbouring points. This procedure can be iterated as many times as necessary until the estimated φ converges to the true solution.

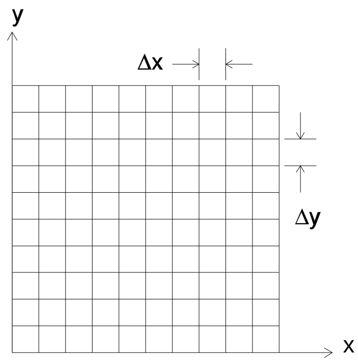

Figure 1: Grid of points in the x, y plane for computing the potential φ.

To relate $\varphi(x, y)$ to $\varphi(x \pm \Delta x, y)$, we can expand φ in a Taylor series to 2nd order as

$$\varphi(x + \Delta x, y) \approx \varphi(x, y) + \Delta x \frac{\partial \varphi}{\partial x}(x, y) + \frac{(\Delta x)^2}{2} \frac{\partial^2 \varphi}{\partial x^2}(x, y) + \ldots \tag{4}$$

$$\varphi(x - \Delta x, y) \approx \varphi(x, y) - \Delta x \frac{\partial \varphi}{\partial x}(x, y) + \frac{(\Delta x)^2}{2} \frac{\partial^2 \varphi}{\partial x^2}(x, y) + \ldots \tag{5}$$

Adding equations (4) and (5) and solving for $\partial^2 \varphi / \partial x^2$ gives

$$\frac{\partial^2 \varphi}{\partial x^2}(x, y) \approx \frac{\varphi(x + \Delta x, y) + \varphi(x - \Delta x, y) - 2\varphi(x, y)}{(\Delta x)^2} \,. \tag{6}$$

The point x, y can be identified by indices i and j using

$$x = i\Delta x, \quad i = 0, \ldots, n_x, \quad a = n_x \Delta x \,, \tag{7}$$

$$y = j\Delta y, \quad j = 0, \ldots, n_y, \quad a = n_y \Delta y \,. \tag{8}$$

Using now the notation φ_{ij} for $\varphi(x, y)$, equation (6) for the second derivative with respect to x becomes

$$\left(\frac{\partial^2 \varphi}{\partial x^2} \right)_{ij} = \frac{\varphi_{i+1,j} + \varphi_{i-1,j} - 2\varphi_{ij}}{(\Delta x)^2} \,. \tag{9}$$

A similar expression for $\partial^2 \varphi / \partial y^2$ can be obtained by switching the two indices. The Laplace equation (with $\partial^2 \varphi / \partial z^2 = 0$) then reads

$$\frac{\varphi_{i+1,j} + \varphi_{i-1,j} - 2\varphi_{ij}}{(\Delta x)^2} + \frac{\varphi_{i,j+1} + \varphi_{i,j-1} - 2\varphi_{ij}}{(\Delta y)^2} = 0 \, . \tag{10}$$

Solving (10) for φ_{ij} gives

$$\varphi_{ij} = \frac{\varphi_{i,j+1} + \varphi_{i,j-1} + r(\varphi_{i+1,j} + \varphi_{i-1,j})}{2(1+r)} \tag{11}$$

where

$$r = \frac{(\Delta x)^2}{(\Delta y)^2} \, . \tag{12}$$

Note that if we take $\Delta x = \Delta y$, then the value of φ at any given grid point is simply the average of its four nearest neighbours.

The idea behind the relaxation method is to use equation (11) iteratively. That is, one starts with a guess for the solution, which could be that $\varphi = \varphi_0$ for $y = a$ and $\varphi = 0$ everywhere else. From this, the right-hand-side of (11) can be evaluated and used to obtained updated values for every point of the grid. That is, the solution at iteration $m+1$ is given in terms of the previous solution m as

$$\varphi_{ij}^{m+1} = \frac{\varphi_{i,j+1}^{m} + \varphi_{i,j-1}^{m} + r(\varphi_{i+1,j}^{m} + \varphi_{i-1,j}^{m})}{2(1+r)} \, . \tag{13}$$

The procedure is continued until the changes in the potential from one iteration to the next become sufficiently small, at which point it is said to have converged. There are various measures of convergence which could be used, such as the largest change out of all the points on the grid, the root mean square change, or the sum of the magnitudes of all the changes. The rate of convergence will also depend on the number of points chosen for the grid.

4 Finding equipotential lines

Once the potential has been found at the various grid points, you should make a graph of the equipotential lines. If you take, say, $\varphi_0 = 100$ Volts, then you will have to find the curves in the x, y plane for which $\varphi = 10, 20, 30, 40, \ldots$ Volts. For any one of these values, say, $\varphi = 30$, you need to find adjacent points on the grid where $\varphi - 30$ changes sign. You can then use linear interpolation to estimate the (x, y) values where $\varphi(x, y) - 30 \approx 0$.

5 Teamwork

You will need to divide up the work so that each member has specific tasks to accomplish and the team achieves its goals to the fullest extent possible. Below are some suggestions on possible ways of separating the project into individual tasks.

In Java you will probably define a class called, say, `Relax` that contains data members for the values of the potential at each of the grid points and methods that perform various operations on the potential. The methods of the `Relax` class should be able to accomplish at least the following tasks:

1. Set up the grid by creating an object of type `Relax`, i.e., in the constructor you should determine the number of grid points, the spacing, boundary conditions, etc.
2. Carry out the updating rule described by equation (13).
3. Test whether the solution has converged.
4. Access the values of the potential.

You can create an `Equipotential` class with methods to calculate equipotential lines, i.e., a set of (x, y) points corresponding to a specified constant value of φ. It will need to be able to take the values of φ for the complete grid as input and return the set of (x, y) points for a set of constant values of the potential so that these can be passed to a plotting routine.

You will need to write a main method that creates the `Relax` object and executes the various methods in the desired way. In addition, someone in the team should investigate the solution described in Sec. 1 by Eqs. (2) and (3). This should be compared with the solution you find with the relaxation method. Providing you have enough time, you should try to investigate problems with more complicated boundary conditions.

6 References

There are a number of good references on the web. The material above is largely adapted from a course page on fluid dynamics at Iowa State University:

`http://www.eng.iastate.edu/~hindman/classes/446/home.html`

A complete program that treats 3-D problems can be found at

`http://www.triumf.ca/relax/doc/relax3d/relax3d.html`

Other web references can be found by searching for '+relaxation +Laplace', etc.

Numerical implementation of the relaxation method is discussed in

N.J. Giordano, *Computational Physics*, Prentice Hall (1997).

G. Cowan, 19 November 2001