Appendix B

The virial theorem

Here we will derive the virial theorem, which describes how the kinetic energy T and potential energy U are shared in a gravitationally bound system. It says that in equilibrium, the mean kinetic energy is related to the mean potential energy by $\langle T \rangle = -\frac{1}{2}\langle U \rangle$. To derive this result, consider a system of masses m_i having position vectors $\mathbf{r}_i$ with $i = 1, \ldots, n$. First we define the quantity A by

$$A = \sum_{i=1}^{n} m_i \dot{\mathbf{r}}_i \cdot \mathbf{r}_i \ , \tag{B.1}$$

where the dot denotes a derivative with respect to time. The time derivative of A is thus

$$\begin{aligned} \dot{A} &= \sum_{i=1}^{n} \left(m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i + m_i \ddot{\mathbf{r}}_i \cdot \mathbf{r}_i \right) \\ &= 2T + \sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{r}_i \ , \end{aligned} \tag{B.2}$$

where to obtain the second line we identified the first term in parentheses as twice the kinetic energy, and in the second term we used Newton's law to replace $m_i \ddot{\mathbf{r}}_i$ by the force on the ith particle, $\mathbf{F}_i$.

If the system is gravitationally bound, then all of the position vectors $\mathbf{r}_i$ remain finite, and therefore A must remain finite. Therefore the average of $\dot{A}$ over a sufficiently long time must go to zero. In this limit we therefore obtain from equation (B.2)

$$\langle 2T \rangle + \left\langle \sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{r}_i \right\rangle = 0 \ . \tag{B.3}$$

This is the general form of the virial theorem.

We can now apply equation (B.3) to the case where our masses interact only by gravity. That is, the force on the mass m_i is given by

$$\mathbf{F}_i = -Gm_i \sum_{\substack{j=1 \\ (j \neq i)}}^{n} m_j \frac{(\mathbf{r}_i - \mathbf{r}_j)}{r_{ij}^3} \, , \tag{B.4}$$

where $r_{ij} \equiv |\mathbf{r}_i - \mathbf{r}_j|$ and the sum over j in (B.4) extends over all of the masses except i, since we are computing the force on m_i. To evaluate equation (B.3) we require the quantity

$$\sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{r}_i = -G \sum_{i=1}^{n} \sum_{\substack{j=1 \\ (j \neq i)}}^{n} \frac{m_i m_j (\mathbf{r}_i - \mathbf{r}_j) \cdot \mathbf{r}_i}{r_{ij}^3} \, . \tag{B.5}$$

The double sum can be broken into those terms where $i < j$ and those where $j < i$:

$$\sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{r}_i = -G \left[\sum_{\substack{i,j \\ (i<j)}} \frac{m_i m_j (\mathbf{r}_i - \mathbf{r}_j) \cdot \mathbf{r}_i}{r_{ij}^3} + \sum_{\substack{i,j \\ (j<i)}} \frac{m_i m_j (\mathbf{r}_i - \mathbf{r}_j) \cdot \mathbf{r}_i}{r_{ij}^3} \right] \, . \tag{B.6}$$

Terms with $i = j$ are of course absent from the sums. Now the second term in (B.6) can be rewritten simply by exchanging the names of the indices i and j to read

$$\sum_{\substack{j,i \\ (i<j)}} \frac{m_j m_i (\mathbf{r}_j - \mathbf{r}_i) \cdot \mathbf{r}_j}{r_{ji}^3} \, . \tag{B.7}$$

As we have $r_{ij} = r_{ji}$, equation (B.6) can therefore be written as

$$\begin{aligned} \sum_{i=1}^{n} \mathbf{F}_i \cdot \mathbf{r}_i &= -G \sum_{\substack{i,j \\ (i<j)}} \frac{m_i m_j (\mathbf{r}_i - \mathbf{r}_j) \cdot (\mathbf{r}_i - \mathbf{r}_j)}{r_{ij}^3} \\ &= -G \sum_{\substack{i,j \\ (i<j)}} \frac{m_i m_j}{r_{ij}} \\ &= U \, . \end{aligned} \tag{B.8}$$

where we identify the second line as the gravitational potential energy U of the system of masses. Notice that the sum with $i < j$ includes each pair of masses once. Combining this with equation (B.3) gives us our final form of the virial theorem,

$$\langle T \rangle = -\tfrac{1}{2} \langle U \rangle \, . \tag{B.9}$$

Although this result has been derived for averages over long periods of time, we will often apply it to an average over a large number of members of a single system considered at a single point in time. The statement that the two are equivalent is called the *ergodic hypothesis*.