Chapter 10

Characteristic Functions

Exercise 10.1: Show that the characteristic function of the Gaussian p.d.f.,

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right) , \tag{10.1}$$

is given by

$$\phi(k) = \exp(i\mu k - \tfrac{1}{2}\sigma^2 k^2) . \tag{10.2}$$

Exercise 10.2: Show that the characteristic function of the exponential p.d.f.,

$$f(x; \xi) = \frac{1}{\xi} e^{-x/\xi}, \tag{10.3}$$

is given by

$$\phi(k) = \frac{1}{1 - ik\xi}. \tag{10.4}$$

Exercise 10.3: Show that the characteristic function of the χ^2 p.d.f. for n degrees of freedom,

$$f(z; n) = \frac{1}{2^{n/2}\Gamma(n/2)} z^{n/2-1} e^{-z/2}, \tag{10.5}$$

is given by

$$\phi(k) = (1 - 2ik)^{-n/2}. \tag{10.6}$$

For this you will need the definition of the gamma function,

$$\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}\, dt. \tag{10.7}$$

Exercise 10.4: Suppose the random variables $x_1, \ldots, x_n$ are independent and each follow a Gaussian distribution of mean μ and variance σ^2. As seen in Chapters 5 and 6, the sample mean

$$\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{10.8}$$

can be used as an estimator for the mean μ.

(a) Find the characteristic function for the sample mean.

(b) From this, show that the p.d.f. for $\overline{x}$ is itself Gaussian, and find its mean and variance.

Exercise 10.5 Using the characteristic function, show that the first four algebraic moments of the Gaussian distribution are

$$\begin{aligned} E[x] &= \mu \\ E[x^2] &= \mu^2 + \sigma^2 \\ E[x^3] &= \mu^3 + 3\mu\sigma^2 \\ E[x^4] &= 3(\mu^2 + \sigma^2)^2. \end{aligned} \tag{10.9}$$

Exercise 10.6: (a) Using the characteristic function, show that the mean and variance of the χ^2 distribution for n degrees of freedom are n and $2n$, respectively.

(b) Suppose z follows the χ^2 distribution for n degrees of freedom. Show that in the limit of large n this becomes a Gaussian distribution with mean $\mu = n$ and variance $\sigma^2 = 2n$. To do this, define the variable

$$y = \frac{z - n}{\sqrt{2n}}\,, \tag{10.10}$$

which has a mean of zero and standard deviation of unity. Show that the characteristic function for y is

$$\phi_y(k) = e^{-ik\sqrt{n/2}} \phi_z \left(\frac{k}{\sqrt{2n}} \right) . \tag{10.11}$$

Expand the logarithm of $\phi_y(k)$ and retain terms that do not vanish in the limit of large n. Transform back to the original variable z to obtain the final result.

Exercise 10.7: Suppose n independent random variables $x_1, \ldots, x_n$ each follow a standard Gaussian distribution, i.e.,

$$\varphi(x_i) = \frac{1}{\sqrt{2\pi}}\, e^{-x_i^2/2} \tag{10.12}$$

for all i, and consider

$$y = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} . \tag{10.13}$$

(a) First consider only one of the x_i. By transformation of variables, show that the p.d.f. of $u = x_i^2$ is

$$f(u) = \frac{1}{\sqrt{2\pi u}}\, e^{-u/2}. \tag{10.14}$$

This is the χ^2 distribution for one degree of freedom.

(b) Show that the characteristic function of u is

$$\phi_u(k) = \frac{1}{\sqrt{1-2ik}}. \tag{10.15}$$

(c) Using the addition theorem, find the characteristic function for

$$v = \sum_{i=1}^{n} x_i^2. \tag{10.16}$$

(d) Using transformation of variables, show that the p.d.f. for $y = (\sum_{i=1}^n x_i^2)^{1/2}$ is

$$h(y) = \frac{1}{2^{n/2-1}\Gamma(n/2)}\, y^{n-1}\, e^{-y/2}. \tag{10.17}$$

This is a special case of the gamma distribution.

(e) Write down the p.d.f. for $n = 3$. This is the Maxwell-Boltzmann distribution. Suppose the velocity components of molecules in a gas v_x, v_y and v_z are independent Gaussian variables with mean values of zero, and standard deviations σ. Write down the p.d.f. for the molecular speed $v = (v_x^2 + v_y^2 + v_z^2)^{1/2}$.

(f) Write down the p.d.f. for $n = 1$. That is, if x follows a standard Gaussian, what is the p.d.f. of $y = |x|$?

Exercise 10.8: Consider a variable x distributed according to the Cauchy (Breit-Wigner) p.d.f.,

$$f(x) \;=\; \frac{1}{\pi}\,\frac{1}{1+x^2}. \tag{10.18}$$

(a) Show that the characteristic function is

$$\phi(k) \;=\; e^{-|k|}. \tag{10.19}$$

(Use the residue theorem and close the integral in the upper half plane for $k > 0$, and in the lower half plane for $k < 0$.)

(b) Consider a sample of n observations of a Cauchy distributed variable x. Using the addition theorem with the characteristic function from (a), show that the sample mean $\overline{x} = \frac{1}{n}\sum_{i=1}^n x_i$ also follows the Cauchy p.d.f. This is a rare case where the p.d.f. of $\overline{x}$ does not change as the sample size increases, and is related to the fact that the moments of the Cauchy distribution does not exist.

Exercise 10.9: The Dirac delta function,

$$f(x;\mu) = \delta(x-\mu), \tag{10.20}$$

is defined by

$$\begin{aligned} \delta(x-\mu) &= 0 \quad \text{for } x \neq \mu\,, \\ \int_{-\infty}^{\infty} \delta(x-\mu)\,dx &= 1. \end{aligned} \tag{10.21}$$

That is, $\delta(x-\mu)$ has an infinitely sharp peak at $x=\mu$ and is zero elsewhere. Find the characteristic function of $\delta(x-\mu)$, and use this to obtain an integral representation of the delta function.