Chapter 11

Unfolding

Exercise 11.1: Consider the detector set-up shown in Fig. 9.1. Suppose the resolution function for x is Gaussian,

$$s(x|x') = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[-\frac{(x-x')^2}{2\sigma_x^2}\right] . \tag{11.1}$$

Find the resolution function for $\cos\theta = x/\sqrt{x^2+d^2}$.

Exercise 11.2: Consider the Tikhonov regularization function with $k = 1$ for a histogram $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_M)$ with equal bin widths,

$$S(\boldsymbol{\mu}) = -\sum_{i=1}^{M-1} (\mu_i - \mu_{i+1})^2 . \tag{11.2}$$

Find the $M \times M$ matrix G such that $S(\boldsymbol{\mu})$ can be expressed in the form

$$S(\boldsymbol{\mu}) \;=\; -\sum_{i,j=1}^{M} G_{ij}\mu_i\mu_j \;=\; -\boldsymbol{\mu}^T G \boldsymbol{\mu} . \tag{11.3}$$

Exercise 11.3: Consider a histogram of expectation values $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_M)$ and the corresponding probabilities $\mathbf{p} = \boldsymbol{\mu}/\mu_{\rm tot}$, where $\mu_{\rm tot} = \sum_{i=1}^{M} \mu_i$.

(a) Show that the Shannon entropy,

$$H(\mathbf{p}) = -\sum_{i=1}^{M} p_i \log p_i , \tag{11.4}$$

is maximum when $p_i = 1/M$ for all i. (Use a Lagrange multiplier to impose the constraint $\sum_{i=1}^{M} p_i = 1$.)

(b) Show that the cross-entropy,

$$K(\mathbf{p};\mathbf{q}) = -\sum_{i=1}^{M} p_i \log \frac{p_i}{M q_i} , \tag{11.5}$$

is maximum when the probabilities $\mathbf{p}$ are equal to the reference distribution $\mathbf{q}$.

Exercise 11.4: Consider an observed histogram $\mathbf{n} = (n_1, \ldots, n_N)$, for which the corresponding expectation values $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_N)$ are related to a true histogram of expectation values $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_M)$ by $\boldsymbol{\nu} = R\boldsymbol{\mu}$. Assume that the covariance matrix V and response matrix R are known and that the histograms contain no background.

(a) Construct estimators for $\boldsymbol{\mu}$ by maximizing

$$\begin{aligned} \Phi(\boldsymbol{\mu}) &= -\frac{\alpha}{2}\chi^2(\boldsymbol{\mu}) + S(\boldsymbol{\mu}) \\ &= -\frac{\alpha}{2}(\mathbf{n} - R\boldsymbol{\mu})^T V^{-1}(\mathbf{n} - R\boldsymbol{\mu}) - \boldsymbol{\mu}^T G\boldsymbol{\mu}\,, \end{aligned} \tag{11.6}$$

where α is the regularization parameter and the $M \times M$ symmetric matrix G is given by a known set of constants (cf. SDA Section 11.5.1). Show that the estimators $\hat{\boldsymbol{\mu}}$ are given by

$$\hat{\boldsymbol{\mu}} = (\alpha R^T V^{-1} R + 2G)^{-1} \alpha R^T V^{-1} \mathbf{n}\,, \tag{11.7}$$

and find the covariance matrix $U_{ij} = \mathrm{cov}[\hat{\mu}_i, \hat{\mu}_j]$.

(b) Now impose the constraint that $\nu_{\mathrm{tot}} = \sum_{i=1}^N \nu_i = \sum_{i=1}^N \sum_{j=1}^M R_{ij}\mu_j$ be equal to the total observed number of events $n_{\mathrm{tot}} = \sum_{i=1}^N n_i$. The solution is found by maximizing

$$\varphi(\boldsymbol{\mu}) = -\frac{\alpha}{2}(\mathbf{n} - R\boldsymbol{\mu})^T V^{-1}(\mathbf{n} - R\boldsymbol{\mu}) - \boldsymbol{\mu}^T G\boldsymbol{\mu} + \lambda(n_{\mathrm{tot}} - \nu_{\mathrm{tot}}) \tag{11.8}$$

with respect to the parameters $\boldsymbol{\mu}$ and the Lagrange multiplier λ. Find the estimators $\hat{\boldsymbol{\mu}}$ and their covariance matrix.

(c) Construct estimators $\hat{\mathbf{b}}$ for the bias $\mathbf{b} = E[\hat{\boldsymbol{\mu}}] - \boldsymbol{\mu}$ using SDA equation (11.76) for both cases (a) and (b).