Chapter 2

Examples of Probability Functions

Exercise 2.1: Consider N multinomially distributed random variables $\mathbf{n} = (n_1, \ldots, n_N)$ with probabilities $\mathbf{p} = (p_1, \ldots, p_N)$ and a total number of trials $n_{\rm tot} = \sum_{i=1}^{N} n_i$. Suppose the variable k is defined as the sum of the first M of the n_i,

$$k = \sum_{i=1}^{M} n_i, \quad M \leq N. \tag{2.1}$$

Use error propagation and the multinomial covariance,

$$\mathrm{cov}[n_i, n_j] = \delta_{ij} n_{\rm tot} p_i (1 - p_i) \; + \; (\delta_{ij} - 1) p_i p_j n_{tot}, \tag{2.2}$$

to find the variance of k. Show that this is equal to the variance of a binomial variable with $p = \sum_{i=1}^{M} p_i$ and $n_{\rm tot}$ trials.

Exercise 2.2: Suppose the random variable x is uniformly distributed in the interval $[\alpha, \beta]$, with $\alpha, \beta > 0$. Find the expectation value of $1/x$, and compare the answer to $1/E[x]$ using $\alpha = 1$, $\beta = 2$.

Exercise 2.3: Consider the exponential p.d.f.,

$$f(x; \xi) = \frac{1}{\xi} \, e^{-x/\xi} \,, \quad x \geq 0 \,. \tag{2.3}$$

(a) Show that the corresponding cumulative distribution is given by

$$F(x) = 1 - e^{-x/\xi} \,, \quad x \geq 0 \,. \tag{2.4}$$

(b) Show that the conditional probability to find a value x between x_0 and $x_0 + x'$ given that $x > x_0$ is equal to the (unconditional) probability to find x less than x', i.e.

$$P(x \leq x_0 + x' | x \geq x_0) = P(x \leq x') \,. \tag{2.5}$$

(c) Cosmic ray muons produced in the upper atmosphere enter a detector at sea level, and some of them come to rest in the detector and decay. The time difference t between entry into the detector and decay follows an exponential distribution, and the mean value of t is the mean lifetime of the muon (approximately 2.2 μS). Explain why the time that the muon lived prior to entering the detector does not play a role in determining the mean lifetime.

Exercise 2.4: Suppose y follows a Gaussian distribution with mean μ and variance σ^2.

(a) Show that

$$x = \frac{y-\mu}{\sigma} \tag{2.6}$$

follows the standard Gaussian $\varphi(x)$ (i.e. having a mean of zero and unit variance).

(b) Show that the cumulative distributions $F(y)$ and $\Phi(x)$ are equal, i.e.

$$F(y) = \Phi\left(\frac{y-\mu}{\sigma}\right). \tag{2.7}$$

Exercise 2.5: (a) Show that if y is Gaussian distributed with mean μ and variance σ^2, then $x = e^y$ follows the log-normal p.d.f.,

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\,\frac{1}{x}\,\exp\left(\frac{-(\log x - \mu)^2}{2\sigma^2}\right). \tag{2.8}$$

(b) Find the expectation value and variance of x by explicitly computing the integrals

$$\begin{aligned} E[x] &= \int x\, f(x;\mu,\sigma^2)\,dx\,, \\ V[x] &= \int (x - E[x])^2\, f(x;\mu,\sigma^2)\,dx\,. \end{aligned} \tag{2.9}$$

(c) Compare the variance from (b) to the approximate result obtained by error propagation with $V[y] = \sigma^2$. Under what conditions is the approximation valid? (Recall that y and hence also σ^2 are dimensionless.)

Exercise 2.6: Show that the cumulative χ^2 distribution for n degrees of freedom can be expressed as

$$F_{\chi^2}(x;n) = P\left(\frac{x}{2},\frac{n}{2}\right), \tag{2.10}$$

where P is the incomplete gamma function,

$$P(n,x) = \frac{1}{\Gamma(n)}\int_0^x e^{-t}\,t^{n-1}\,dt\,. \tag{2.11}$$