Chapter 9

Statistical Errors, Confidence Intervals and Limits

Exercise 9.1: Suppose an estimator $\hat{\theta}$ is Gaussian distributed about the parameter's true value θ with a standard deviation $\sigma_{\hat{\theta}}$. Assume that $\sigma_{\hat{\theta}}$ is known.

(a) Sketch the functions $u_\alpha(\theta)$ and $v_\beta(\theta)$ defining the confidence belt (cf. SDA Section 9.2).

(b) Show that the central confidence interval for θ at a confidence level $1-\gamma$ is given by

$$[\hat{\theta} - \sigma_{\hat{\theta}}\Phi^{-1}(1-\gamma/2),\, \hat{\theta} + \sigma_{\hat{\theta}}\Phi^{-1}(1-\gamma/2)], \tag{9.1}$$

where Φ^{-1} is the quantile of the standard Gaussian.

Exercise 9.2: (a) Consider n observations of an exponentially distributed variable x with mean ξ. The ML estimator for ξ (see SDA (6.6)) is given by

$$\hat{\xi} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{9.2}$$

and the p.d.f. for $\hat{\xi}$ (cf. SDA equation (10.25)) is

$$g(\hat{\xi};\xi) \;=\; \frac{n^n}{(n-1)!}\frac{\hat{\xi}^{n-1}}{\xi^n}e^{-n\hat{\xi}/\xi}. \tag{9.3}$$

(a) Show that the curves defining the confidence belt, $u_\alpha(\xi)$ and $v_\beta(\xi)$, are given by

$$\begin{aligned} u_\alpha(\xi) &= \frac{\xi}{2n}F^{-1}_{\chi^2}(1-\alpha;2n), \\ v_\beta(\xi) &= \frac{\xi}{2n}F^{-1}_{\chi^2}(\beta;2n), \end{aligned} \tag{9.4}$$

where $F^{-1}_{\chi^2}$ is the quantile of the χ^2 distribution. Use the fact that the cumulative χ^2 distribution can be related to the incomplete gamma function $P(n,x)$ by (cf. Exercise 2.5)

$$F_{\chi^2}(2x; 2n) \; = \; P(n, x) \; \equiv \; \int_0^x e^{-t} t^{n-1} dt. \tag{9.5}$$

Make a sketch of $u_\alpha(\xi)$ and $v_\beta(\xi)$ using $\alpha = \beta = 0.159$ and $n = 5$. (Quantiles of the χ^2 distribution can be looked up in standard tables or obtained from the routine `CHISIN` from the CERN program library.)

(b) Find the confidence interval $[a, b]$ as a function of the estimate $\hat{\xi}$, the sample size n and the confidence levels α and β. Suppose the estimate is $\hat{\xi} = 1.0$. Sketch this on the plot of $u_\alpha(\xi)$ and $v_\beta(\xi)$. Evaluate a and b for $n = 5$, $\alpha = \beta = 0.159$. Compare the result to the interval obtained from plus or minus one standard deviation about the estimate $\hat{\xi}$.

Exercise 9.3: Show that the upper and lower limits for the parameter p of a binomial distribution are

$$\begin{aligned} p_{\rm lo} &= \frac{n F_F^{-1}[\alpha; 2n, 2(N-n+1)]}{N-n+1 \; + \; n F_F^{-1}[\alpha; 2n, 2(N-n+1)]} \\ p_{\rm up} &= \frac{(n+1) F_F^{-1}[1-\beta; 2(n+1), 2(N-n)]}{(N-n) \; + \; (n+1) F_F^{-1}[1-\beta; 2(n+1), 2(N-n)]} \, . \end{aligned} \tag{9.6}$$

Here the confidence levels for the upper and lower limits are $1-\alpha$ and $1-\beta$, respectively, n is the number of successes observed in N trials, and F_F^{-1} is the quantile of the F distribution. This is defined by the p.d.f.

$$f(x; n_1, n_2) = \left(\frac{n_1}{n_2} \right)^{n_1/2} \frac{\Gamma(\frac{1}{2}(n_1+n_2))}{\Gamma(\frac{1}{2}n_1)\Gamma(\frac{1}{2}n_2)} \, x^{n_1/2-1} \left(1 + \frac{n_1}{n_2} x \right)^{-(n_1+n_2)/2} , \tag{9.7}$$

where $x > 0$ and n_1 and n_2 are integer parameters (degrees of freedom). Use the fact that the cumulative binomial distribution is related to the cumulative distribution $F_F(x)$ for $n_1 = 2(n+1)$ and $n_2 = 2(N-n)$ degrees of freedom by [1]

$$\sum_{k=0}^{n} \frac{N!}{k!(N-k)!} \, p^k \, (1-p)^{N-k} \; = \; 1 \; - \; F_F \left[\frac{(N-n)p}{(n+1)(1-p)}; 2(n+1), 2(N-n) \right] . \tag{9.8}$$

Quantiles of the F distribution can be obtained from standard tables or computed with the routine `ffinv`. Equations (9.6) are implemented in the routines `binomlo`, `binomup` and `binomint`.

Exercise 9.4: In an antineutrino-nucleon scattering experiment with the Gargamelle bubble chamber at CERN, events were selected having only hadrons (from the neutral-current proccess $\overline{\nu}_\mu N \to \overline{\nu}_\mu X$) or with hadrons and a muon (the charged-current procces $\overline{\nu}_\mu N \to \mu^+ X$). Out of a sample of 212 events, 64 were classified as neutral current (NC) and 148 as charged current (CC). Estimate the probability for an event to be NC and find the 68.3% central confidence

[1] Use of the F distribution for evaluating binomial confidence intervals is due to A. Hald, *Statistical Theory with Engineering Applications*, John Wiley, New York, 1952.

interval. Find the corresponding estimator and interval for the ratio of probabilities for NC and CC events.[2]

Exercise 9.5: In an experiment investigating particle collisions, 10 events are selected as being of a certain type, say, having a high value of some property x. Out of the 10 high-x events, 2 are found to contain muons.

(a) Find the 68.3% central confidence interval for the binomial parameter p for high-x events to contain muons. Express the answer as $p = \hat{p}^{+c}_{-d}$ where $\hat{p}$ is the ML estimate for p and $[\hat{p}-c, \hat{p}+d]$ is the confidence interval. (The routine `binomint` can be used.)

(b) Compare the interval from (a) to $\hat{p} \pm \hat{\sigma}_{\hat{p}}$, where $\hat{\sigma}_{\hat{p}}$ is the estimate of the standard deviation of $\hat{p}$.

(c) A common mistake is to regard the number 10 of high-x events as a random variable and to include its variance in the error for $\hat{p}$ (e.g. using error propagation). Why is this not the correct approach?

Exercise 9.6: Suppose that to produce the events in Exercise 9.5, the total amount of data collected corresponded to an integrated luminosity of $L = 1\ \mathrm{pb}^{-1}$ (known with a negligible error). The total number of events produced of a given type can be regarded as a Poisson variable n with mean value $\nu = \sigma L$, where σ is the production cross section. (Why is the Poisson distribution appropriate?)

(a) Find the 68.3% central confidence intervals for the expected numbers of events ν_x and $\nu_{x\mu}$ for events with high x and high x with muons, given $n_x = 10$ and $n_{x\mu} = 2$ events observed. What are the corresponding confidence intervals for the production cross sections, σ_x and $\sigma_{x\mu}$?

(b) Compare the confidence intervals from (a) to intervals constructed as plus or minus one standard deviation about the estimate of the corresponding parameter.

(c) Suppose that in a separate experiment with an integrated luminosity of $L' = 100\ \mathrm{pb}^{-1}$, $n'_x = 1173$ high-x events are observed. This experiment, however, is unable to identify muons. Construct the log-likelihood function for the parameters σ_x and $p = \sigma_{x\mu}/\sigma_x$ using the data n_x, $n_{x\mu}$ and n'_x. Show that the ML estimator for p does not depend on n'_x. Does it make sense that the result of the second experiment has no impact on the estimate of p?

(d) Suppose the original experiment had not measured the number of high-x events but had only reported the number of high-x events with muons. Using only the two results $n_{x\mu} = 2$ and $n'_x = 1173$, construct the log-likelihood function for σ_x and p. From this find the ML estimators. Use error propagation to estimate the standard deviation of $\hat{p}$, and compare the interval $\hat{p} \pm \hat{\sigma}_{\hat{p}}$ to the intervals from Exercises 9.5 (a) and (b). Optional: How would you go about constructing a confidence interval for p in this case?

Exercise 9.7: A Particle created in an interaction is emitted at a certain angle with respect to the z axis, as shown in Fig. 9.1. A detector located a distance d from the interaction point measures the particle's position x perpendicular to the z direction. The angle θ is defined as the angle between the z axis and the projection of the particle's trajectory into the (x, z) plane. Suppose the measured value x can be regarded as a Gaussian variable centered about the true value and having a standard deviation σ_x.

[2] In the actual experiment, small background corrections were included; see F.J. Hasert et al., Observation of neutrino-like interactions without muon or electron in the Gargamelle neutrino experiment, *Phys. Lett.* **46B** (1973) 138.

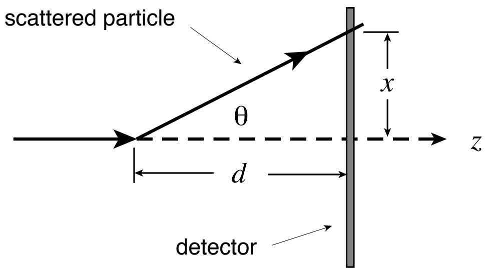

Figure 9.1: The definition of the scattering angle θ from the trajectory projected into the (x, z) plane.

(a) Find the central confidence interval at a confidence level $1 - \gamma$ for the cosine of the angle θ. Assume that the distance d is known without error.

(b) Take $d = 1$ m, $\sigma_x = 1$ mm and suppose the measured value is $x = 2.0$ mm. Find the 68.3% and 95% central confidence intervals for $\cos\theta$.