Statistical Methods for Particle Physics
Graduierten-Kolleg RWTH Aachen, February 2014
Problem sheet 5

Exercise 1: The binomial distribution is given by

$$f(n; N, \theta) = \frac{N!}{n!(N-n)!}\theta^n (1-\theta)^{N-n} ,$$

where n is the number of 'successes' in N independent trials, with a success probability of θ for each trial. Recall that the expectation value and variance of n are $E[n] = N\theta$ and $V[n] = N\theta(1-\theta)$, respectively. Suppose we have a single observation of n and using this we want to estimate the parameter θ.

1. Find the maximum likelihood estimator $\hat{\theta}$.
2. Show that $\hat{\theta}$ has zero bias and find its variance.
3. Suppose we observe $n = 0$ for $N = 10$ trials. Find the upper limit for θ at a confidence level of CL = 95% and evaluate numerically.
4. Suppose we treat the problem with the Bayesian approach using the Jeffreys prior, $\pi(\theta) \propto \sqrt{I(\theta)}$, where

   $$I(\theta) = -E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right]$$

   is the expected Fisher information. Find the Jeffreys prior $\pi(\theta)$ and the posterior pdf $p(\theta|n)$ as proportionalities.
5. Explain how in the Bayesian approach how one would determine an upper limit on θ using the result from (d). (You do not actually have to calculate the upper limit.)

   Explain briefly the differences in the interpretation between frequentist and Bayesian upper limits.

Exercise 2: Consider the pdf for the continuous random variable x

$$f(x; \theta) = \frac{x}{\theta^2} e^{-x/\theta} ,$$

where $0 \leq x < \infty$ and $\theta > 0$. The expectation value and variance of x are $E[x] = 2\theta$ and $V[x] = 2\theta^2$. Suppose we have independent values $x_1, \ldots, x_n$ sampled from this pdf.

2(a): Show that the maximum likelihood estimator for θ is

$$\hat{\theta} = \frac{\overline{x}}{2} ,$$

where $\overline{x}$ is the arithmetic average of the x values.

2(b): Find the estimator's bias and variance using the information given above for the expectation value and variance of x.

2(c) Find the minimum variance bound (MVB) of the estimator.

2(d) Make a sketch of the log-likelihood function indicating the estimator $\hat{\theta}$ and show on the sketch now to find the standard deviation of $\hat{\theta}$ graphically.

2(e) Suppose how that the number of x values, n, is regarded not as constant but as following a Poisson distribution with an expectation value $E[n] = c/\theta$, where c is a given constant. Find the (extended) maximum likelihood estimator for θ. (Recall the Poisson distribution for a mean value ν is given by $P(n;\nu) = \nu^n e^{-\nu}/n!$.)