# Introduction to Statistical Methods for High Energy Physics

2005 CERN Summer Student Lectures

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- CERN course web page:

  www.pp.rhul.ac.uk/~cowan/stat_cern.html

- See also University of London course web page:

  www.pp.rhul.ac.uk/~cowan/stat_course.html

# Outline

## Lecture 1

Probability
Random variables, probability densities, etc.
Brief catalogue of probability densities

## Lecture 2

The Monte Carlo method
Statistical tests
Fisher discriminants, neural networks, etc.

## Lecture 3

Parameter estimation
The method of maximum likelihood
The method of least squares
Interval estimation (setting limits)

# Some statistics books, papers, etc.

G. Cowan, *Statistical Data Analysis*, Clarendon, Oxford, 1998

see also `www.pp.rhul.ac.uk/~cowan/sda`

R.J. Barlow, Statistics, A Guide to the Use of Statistical in the Physical Sciences, Wiley, 1989

see also `hepwww.ph.man.ac.uk/~roger/book.html`

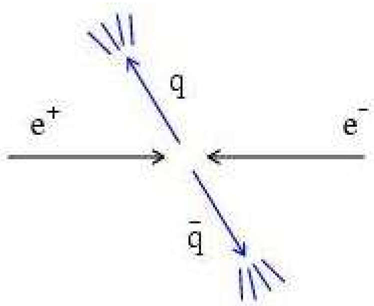
L. Lyons, *Statistics for Nuclear and Particle Physics*, CUP, 1986

W. Eadie et al., *Statistical and Computational Methods in Experimental Physics*, North-Holland, 1971

S. Brandt, *Statistical and Computational Methods in Data Analysis*, Springer, New York, 1998 (with program library on CD)

S. Eidelman et al. (Particle Data Group), *Review of Particle Physics*, Phys. Lett. B592 (2004) 1; see also `pdg.lbl.gov` sections on probability statistics, Monte Carlo

# Data analysis in particle physics




Observe events of a certain type

Measure characteristics of each event (particle momenta, number of muons, energy of jets,...)

Theories (e.g. SM) predict distributions of these properties up to free parameters, e.g., $\alpha$, $G_F$, $M_Z$, $\alpha_s$, $m_H$, ...

Some tasks of data analysis:

Estimate (measure) the parameters;

Quantify the uncertainty of the parameter estimates;

Test the extent to which the predictions of a theory are in agreement with the data.

# Dealing with uncertainty

In particle physics there are various elements of uncertainty:

theory is not deterministic

quantum mechanics

random measurement errors

present even without quantum effects

things we could know in principle but don’t

e.g. from limitations of cost, time, ...

We can quantify the uncertainty using PROBABILITY

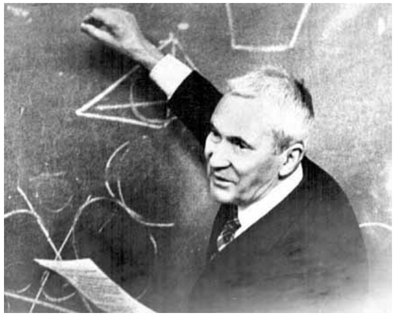
# A definition of probability

Consider a set $S$ with subsets $A$, $B$, ...

$$\text{For all } A \subset S, P(A) \geq 0$$

$$P(S) = 1$$

$$\text{If } A \cap B = \emptyset, P(A \cup B) = P(A) + P(B)$$



Kolmogorov axioms (1933)

From these axioms we can derive further properties, e.g.

$$P(\overline{A}) = 1 - P(A)$$

$$P(A \cup \overline{A}) = 1$$

$$P(\emptyset) = 0$$

$$\text{if } A \subset B, \text{ then } P(A) \leq P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

# Conditional probability, independence

Also define conditional probability of $A$ given $B$ (with $P(B) \neq 0$):

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

E.g. rolling dice: $P(n < 3 \,|\, n \text{ even}) = \frac{P((n<3) \cap n \text{ even})}{P(\text{even})} = \frac{1/6}{3/6} = \frac{1}{3}$

Subsets $A$, $B$ independent if: $P(A \cap B) = P(A)P(B)$

If $A$, $B$ independent, $P(A|B) = \frac{P(A)P(B)}{P(B)} = P(A)$

N.B. do not confuse with disjoint subsets, i.e., $A \cap B = \emptyset$

# Interpretation of probability

## I. Relative frequency

$A$, $B$, ... are outcomes of a repeatable experiment

$$P(A) = \lim_{n\to\infty} \frac{\text{times outcome is } A}{n}$$

cf. quantum mechanics, particle scattering, radioactive decay...

## II. Subjective probability

$A$, $B$, ... are hypotheses (statements that are true or false)

$$P(A) = \text{degree of belief that } A \text{ is true}$$

- Both interpretations consistent with Kolmogorov axioms.
- In particle physics frequency interpretation often most useful, but subjective probability can provide more natural treatment of non-repeatable phenomena:

  systematic uncertainties, probability that Higgs boson exists,...

# Bayes' theorem

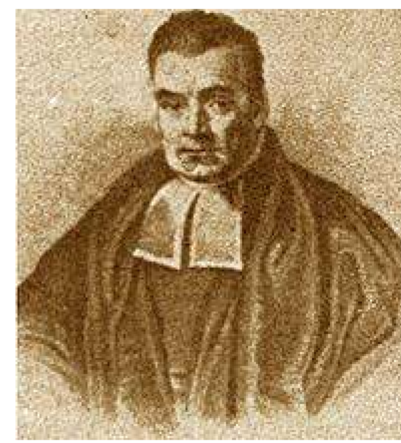
From the definition of conditional probability we have,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

but $P(A \cap B) = P(B \cap A)$, so

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

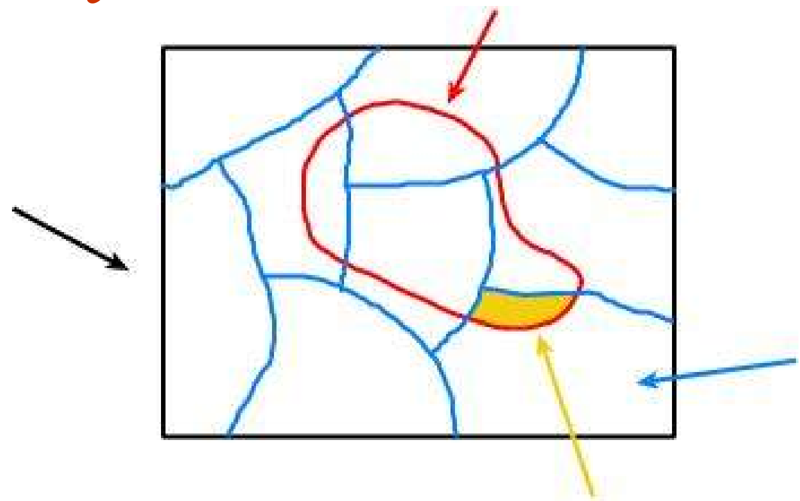
Bayes' theorem



First published (posthumously) by the Reverend Thomas Bayes (1702−1761)

*An essay towards solving a problem in the doctrine of chances*, Philos. Trans. R. Soc. **53** (1763) 370; reprinted in Biometrika, **45** (1958) 293.

# The law of total probability




Consider a subset $B$ of the sample space $S$,

divided into disjoint subsets $A_i$ such that $\cup_i A_i = S$,

$$\rightarrow \quad B = B \cap S = B \cap (\cup_i A_i) = \cup_i (B \cap A_i),$$

$$\rightarrow \quad P(B) = P(\cup_i (B \cap A_i)) = \textstyle\sum_i P(B \cap A_i)$$

$$\rightarrow \quad P(B) = \textstyle\sum_i P(B|A_i) P(A_i)$$ law of total probability

Bayes' theorem becomes

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{\sum_i P(B|A_i)P(A_i)}}$$

# An example using Bayes' theorem

Suppose the probability (for anyone) to have AIDS is:

$$P(\text{AIDS}) = 0.001$$
$$P(\text{no AIDS}) = 0.999$$

← prior probabilities, i.e., before any test carried out

Consider an AIDS test: result is + or –

$$P(+|\text{AIDS}) = 0.98$$
$$P(-|\text{AIDS}) = 0.02$$

← probabilities to (in)correctly identify an infected person

$$P(+|\text{no AIDS}) = 0.03$$
$$P(-|\text{no AIDS}) = 0.97$$

← probabilities to (in)correctly identify an uninfected person

Suppose your result is +. How worried should you be?

# Bayes' theorem example (cont.)

The probability to have AIDS given a + result is

$$P(\text{AIDS}|+) = \frac{P(+|\text{AIDS})P(\text{AIDS})}{P(+|\text{AIDS})P(\text{AIDS}) + P(+|\text{no AIDS})P(\text{no AIDS})}$$

$$= \frac{0.98 \times 0.001}{0.98 \times 0.001 + 0.03 \times 0.999}$$

$$= 0.032 \qquad \leftarrow \text{posterior probability}$$

i.e. you're probably OK!

Your viewpoint: my degree of belief that I have AIDS is 3.2%

Your doctor's viewpoint: 3.2% of people like this will have AIDS

# Frequentist Statistics − general philosophy

In frequentist statistics, probabilities are associated only with the data, i.e., outcomes of repeatable observations (shorthand: $\vec{x}$).

Probability = limiting frequency

Probabilities such as

$P$ (Higgs boson exists),
$P\,(0.117 < \alpha_s < 0.121)$,

etc. are either 0 or 1, but we don't know which.
The tools of frequentist statistics tell us what to expect, under the assumption of certain probabilities, about hypothetical repeated observations.

The preferred theories (models, hypotheses, ...) are those for which our observations would be considered 'usual'.

# Bayesian Statistics – general philosophy

In Bayesian statistics, use subjective probability for hypotheses:

probability of the data assuming hypothesis $H$ (the likelihood)

prior probability, i.e., before seeing the data

$$P(H|\vec{x}) = \frac{P(\vec{x}|H)\pi(H)}{\int P(\vec{x}|H)\pi(H)\,dH}$$

posterior probability, i.e., after seeing the data

normalization involves sum over all possible hypotheses

Bayes' theorem has an "if-then" character: If your prior probabilities were $\pi(H)$, then it says how these probabilities should change in the light of the data.

No general prescription for priors (subjective!)

# Random variables and probability density functions

A random variable is a numerical characteristic assigned to an element of the sample space; can be discrete or continuous.

Suppose outcome of experiment is continuous value $x$

$$P(x \text{ found in } [x, x+dx]) = f(x)\,dx$$

→ $f(x)$ = probability density function (pdf)

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$x$ must be somewhere

Or for discrete outcome $x_i$ with e.g. $i = 1, 2, \ldots$ we have

$$P(x_i) = p_i$$

probability mass function

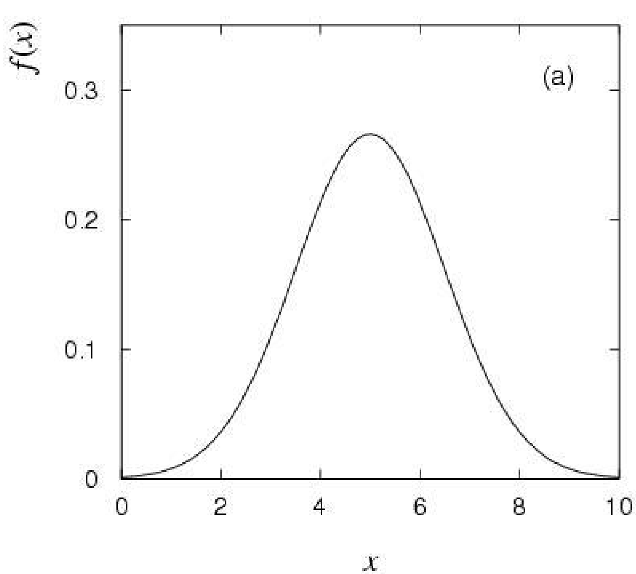
$$\sum_i P(x_i) = 1$$

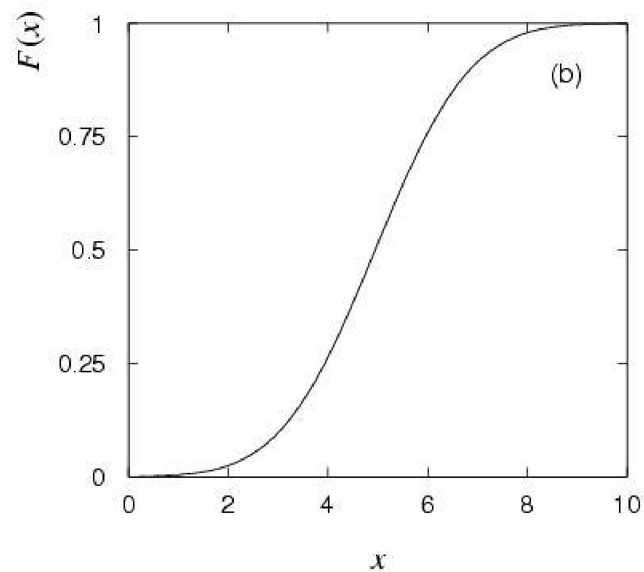
$x$ must take on one of its possible values

# Cumulative distribution function

Probability to have outcome less than or equal to $x$ is

$$\int_{-\infty}^{x} f(x')\,dx' \equiv F(x)$$ cumulative distribution function




Alternatively define pdf with $f(x) = \dfrac{\partial F(x)}{\partial x}$

# Histograms

pdf = histogram with

infinite data sample,

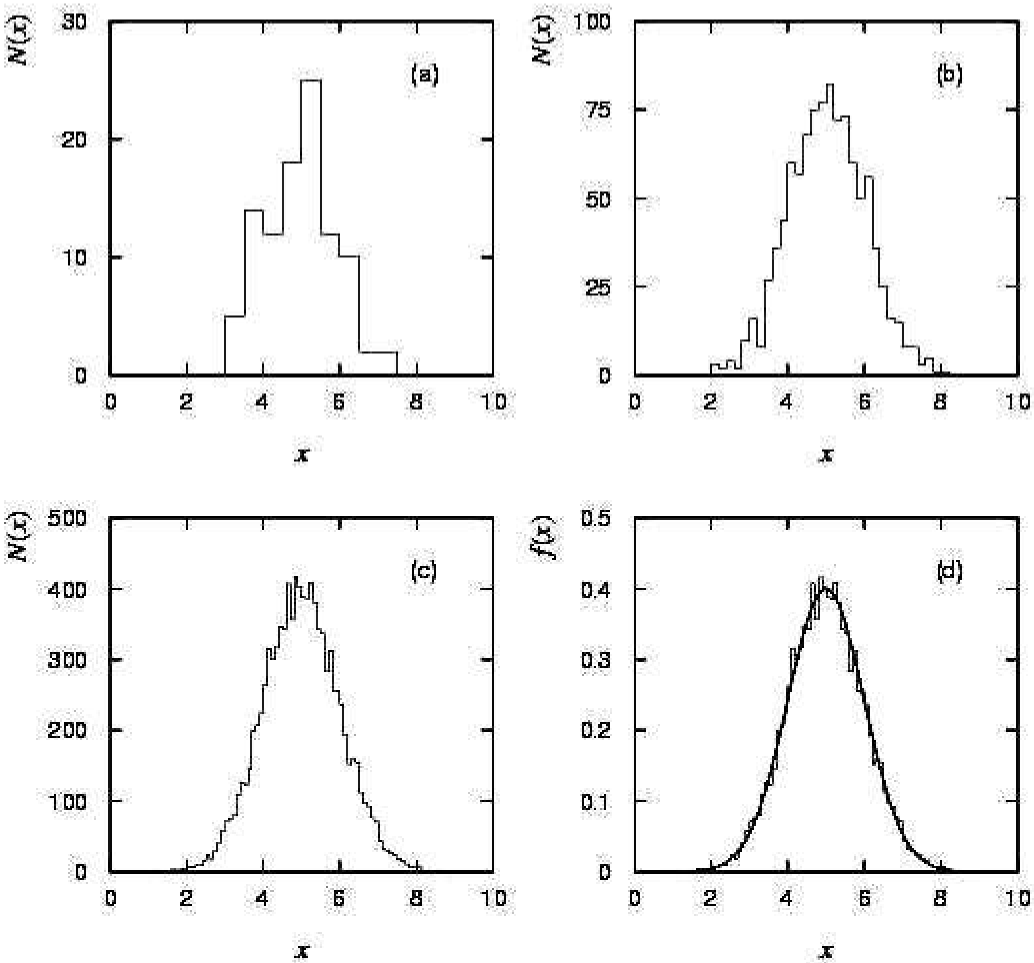
zero bin width,

normalized to unit area.

$$f(x) = \frac{N(x)}{n\Delta x}$$

$n =$ number of entries

$\Delta x =$ bin width



# Other types of probability densities

Outcome of experiment characterized by several values, e.g. an $n$-component vector, $(x_1, \ldots x_n)$

→ joint pdf $f(x_1, \ldots, x_n)$

Sometimes we want only pdf of some (or one) of the components

→ marginal pdf $f_1(x_1) = \int \cdots \int f(x_1, \ldots, x_n)\, dx_2 \ldots dx_n$

$x_1, x_2$ independent if $f(x_1, x_2) = f_1(x_1) f_2(x_2)$

Sometimes we want to consider some components as constant

→ conditional pdf $g(x_1 | x_2) = \dfrac{f(x_1, x_2)}{f_2(x_2)}$

# Expectation values

Consider continuous r.v. $x$ with pdf $f(x)$.

Define expectation (mean) value as $E[x] = \int x\, f(x)\, dx$

Notation (often): $E[x] = \mu$ ~ "centre of gravity" of pdf.

For a function $y(x)$ with pdf $g(y)$,

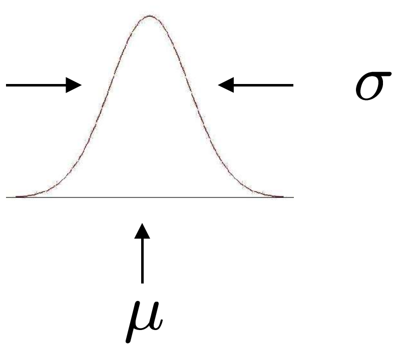
$$E[y] = \int y\, g(y)\, dy = \int y(x) f(x)\, dx \qquad \text{(equivalent)}$$

Variance: $V[x] = E[x^2] - \mu^2 = E[(x-\mu)^2]$

Notation: $V[x] = \sigma^2$

Standard deviation: $\sigma = \sqrt{\sigma^2}$

$\sigma$ ~ width of pdf, same units as $x$.



# Covariance and correlation

Define covariance cov[$x$,$y$] (also use matrix notation $V_{xy}$) as

$$\mathrm{cov}[x, y] = E[xy] - \mu_x \mu_y = E[(x - \mu_x)(y - \mu_y)]$$

Correlation coefficient (dimensionless) defined as

$$\rho_{xy} = \frac{\mathrm{cov}[x, y]}{\sigma_x \sigma_y}$$

If $x$, $y$, independent, i.e., $f(x, y) = f_x(x) f_y(y)$ , then

$$E[xy] = \int\int xy\, f(x, y)\, dx dy = \mu_x \mu_y$$

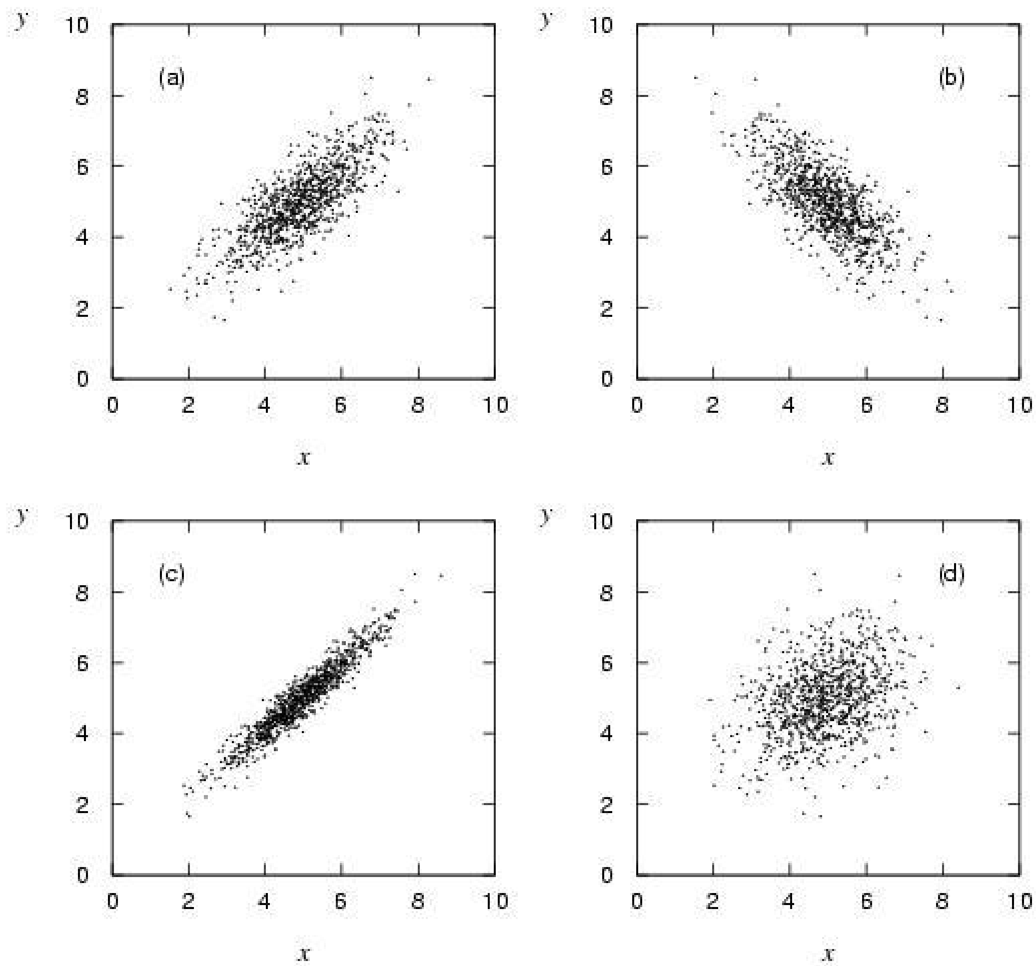
$\rightarrow$ $\mathrm{cov}[x, y] = 0$ $x$ and $y$, 'uncorrelated'

N.B. converse not always true.

# Correlation (cont.)

$\rho = 0.75$

$\rho = -0.75$




$\rho = 0.95$

$\rho = 0.25$

# Error propagation

Suppose we measure a set of values $\vec{x} = (x_1, \ldots, x_n)$

and we have the covariances $V_{ij} = \mathrm{cov}[x_i, x_j]$

which quantify the measurement errors in the $x_i$.

Now consider a function $y(\vec{x})$ .

What is the variance of $y(\vec{x})$ ?

The hard way: use joint pdf $f(\vec{x})$ to find the pdf $g(y)$ ,

then from $g(y)$ find $V[y] = E[y^2] - (E[y])^2$.

Often not practical, $f(\vec{x})$ may not even be fully known.

# Error propagation (2)

Suppose we had $\vec{\mu} = E[\vec{x}]$

in practice only estimates given by the measured $\vec{x}$

Expand $y(\vec{x})$ to 1st order in a Taylor series about $\vec{\mu}$

$$y(\vec{x}) \approx y(\vec{\mu}) + \sum_{i=1}^{n} \left[ \frac{\partial y}{\partial x_i} \right]_{\vec{x}=\vec{\mu}} (x_i - \mu_i)$$

To find $V[y]$ we need $E[y^2]$ and $E[y]$.

$$E[y(\vec{x})] \approx y(\vec{\mu}) \quad \text{since} \quad E[x_i - \mu_i] = 0$$

# Error propagation (3)

$$E[y^2(\vec{x})] \approx y^2(\vec{\mu}) \; + \; 2y(\vec{\mu}) \sum_{i=1}^{n} \left[ \frac{\partial y}{\partial x_i} \right]_{\vec{x}=\vec{\mu}} E[x_i - \mu_i]$$

$$+E\left[ \left( \sum_{i=1}^{n} \left[ \frac{\partial y}{\partial x_i} \right]_{\vec{x}=\vec{\mu}} (x_i - \mu_i) \right) \left( \sum_{j=1}^{n} \left[ \frac{\partial y}{\partial x_j} \right]_{\vec{x}=\vec{\mu}} (x_j - \mu_j) \right) \right]$$

$$= y^2(\vec{\mu}) + \sum_{i,j=1}^{n} \left[ \frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} \right]_{\vec{x}=\vec{\mu}} V_{ij}$$

Putting the ingredients together gives the variance of $y(\vec{x})$

$$\boxed{\sigma_y^2 \approx \sum_{i,j=1}^{n} \left[ \frac{\partial y}{\partial x_i} \frac{\partial y}{\partial x_j} \right]_{\vec{x}=\vec{\mu}} V_{ij}}$$

# Error propagation (4)

If the $x_i$ are uncorrelated, i.e., $V_{ij} = \sigma_i^2 \delta_{ij}$ , then this becomes

$$\sigma_y^2 \approx \sum_{i=1}^{n} \left[ \frac{\partial y}{\partial x_i} \right]^2_{\vec{x}=\vec{\mu}} V_{ij}$$

Similar for a set of $m$ functions $\vec{y}(\vec{x}) = (y_1(\vec{x}), \ldots, y_m(\vec{x}))$

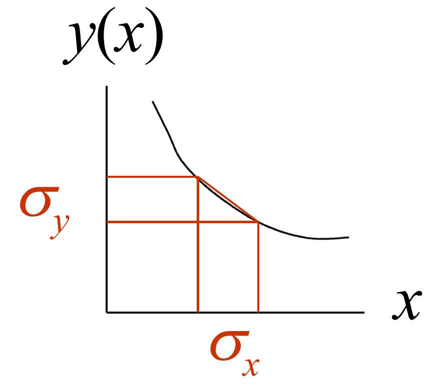
$$\boxed{U_{kl} = \mathrm{cov}[y_k, y_l] \approx \sum_{i,j=1}^{n} \left[ \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} \right]_{\vec{x}=\vec{\mu}} V_{ij}}$$

or in matrix notation $U = AVA^T$ , where

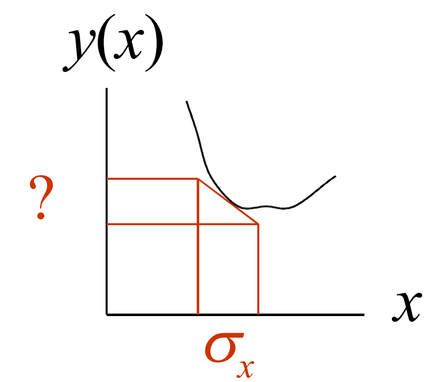
$$A_{ij} = \left[ \frac{\partial y_i}{\partial x_j} \right]_{\vec{x}=\vec{\mu}}$$

# Error propagation (5)

The 'error propagation' formulae tell us the covariances of a set of functions $\vec{y}(\vec{x}) = (y_1(\vec{x}), \ldots, y_m(\vec{x}))$ in terms of the covariances of the original variables.




Limitations: exact only if $\vec{y}(\vec{x})$ linear.
Approximation breaks down if function nonlinear over a region comparable in size to the $\sigma_i$.




N.B. We have said nothing about the exact pdf of the $x_i$, e.g., it doesn't have to be Gaussian.

# Error propagation – special cases

$$y = x_1 + x_2 \quad \rightarrow \quad \sigma_y^2 = \sigma_1^2 + \sigma_2^2 + 2\,\mathrm{cov}[x_1, x_2]$$

$$y = x_1 x_2 \quad \rightarrow \quad \frac{\sigma_y^2}{y^2} = \frac{\sigma_1^2}{x_1^2} + \frac{\sigma_2^2}{x_2^2} + 2\frac{\mathrm{cov}[x_1, x_2]}{x_1 x_2}$$

That is, if the $x_i$ are uncorrelated:

add errors quadratically for the sum (or difference),

add relative errors quadratically for product (or ratio).

But correlations can change this completely...

# Error propagation – special cases (2)

Consider $y = x_1 - x_2$ with

$$\mu_1 = \mu_2 = 10, \quad \sigma_1 = \sigma_2 = 1, \quad \rho = \frac{\mathrm{cov}[x_1, x_2]}{\sigma_1 \sigma_2} = 0\,.$$

$$V[y] = 1^2 + 1^2 = 2, \rightarrow \sigma_y = 1.4$$

Now suppose $\rho = 1$. Then

$$V[y] = 1^2 + 1^2 - 2 = 0, \rightarrow \sigma_y = 0$$

i.e. for 100% correlation, error in difference → 0.

# Some distributions

| Distribution/pdf | Example use in HEP |
| --- | --- |
| Binomial | Branching ratio |
| Multinomial | Histogram with fixed $N$ |
| Poisson | Number of events found |
| Uniform | Monte Carlo method |
| Exponential | Decay time |
| Gaussian | Measurement error |
| Chi-square | Goodness-of-fit |
| Cauchy | Mass of resonance |
| Landau | Ionization energy loss |

# Binomial distribution

Consider $N$ independent experiments (Bernoulli trials):

outcome of each is 'success' or 'failure',

probability of success on any given trial is $p$.

Define discrete r.v. $n$ = number of successes ($0 \leq n \leq N$).

Probability of a specific outcome (in order), e.g. 'ssfsf' is

$$pp(1-p)p(1-p) = p^n(1-p)^{N-n}$$

But order not important; there are $\dfrac{N!}{n!(N-n)!}$

ways (permutations) to get $n$ successes in $N$ trials, total probability for $n$ is sum of probabilities for each permutation.

# Binomial distribution  (2)

The binomial distribution is therefore

$$f(n; N, p) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}$$

random variable

parameters

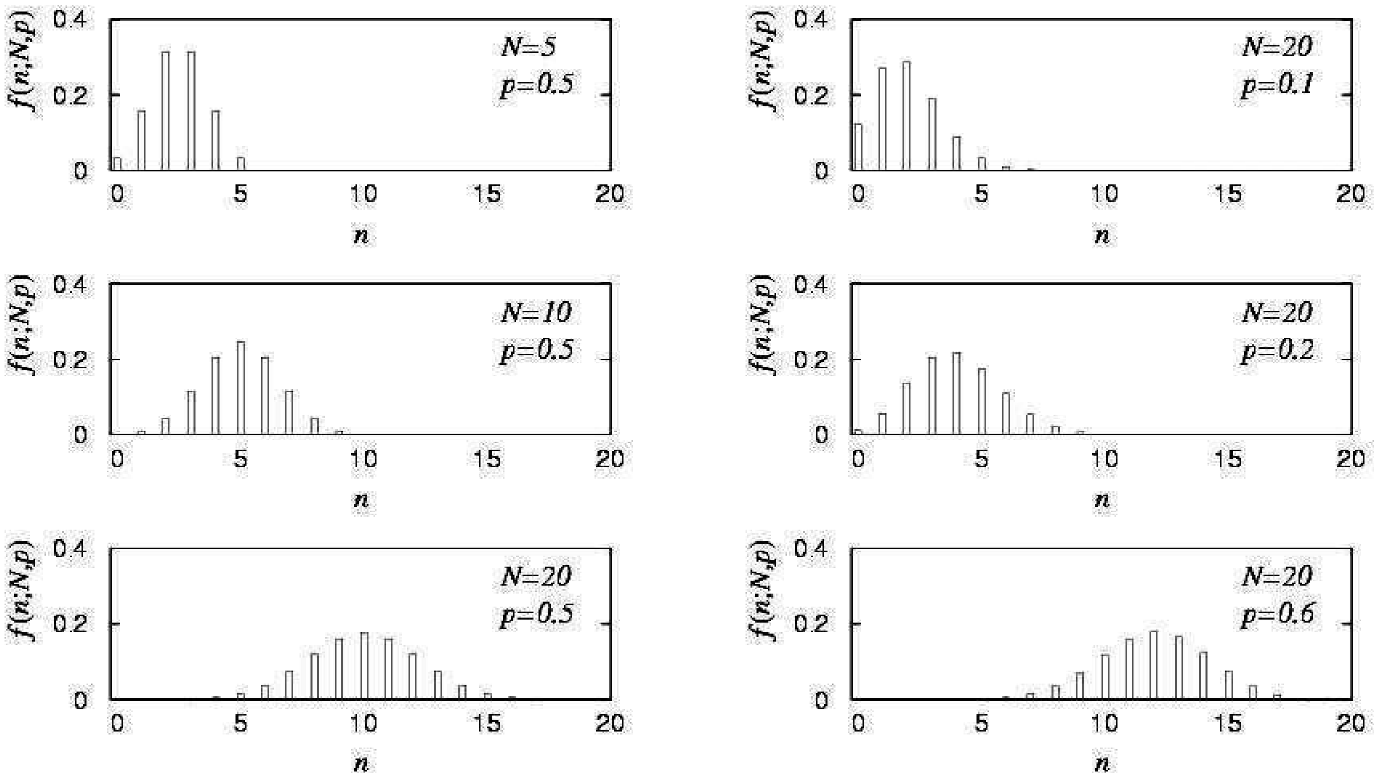
For the expectation value and variance we find:

$$E[n] = \sum_{n=0}^{N} n f(n; N, p) = Np$$

$$V[n] = E[n^2] - (E[n])^2 = Np(1-p)$$

# Binomial distribution (3)

Binomial distribution for several values of the parameters:




Example: observe $N$ decays of $W^{\pm}$, the number $n$ of which are $W \to \mu\nu$ is a binomial r.v., $p$ = branching ratio.

# Multinomial distribution

Like binomial but now $m$ outcomes instead of two, probabilities are

$$\vec{p} = (p_1, \ldots, p_m)\,, \quad \text{with} \ \sum_{i=1}^{m} p_i = 1\,.$$

For $N$ trials we want the probability to obtain:

$n_1$ of outcome 1,

$n_2$ of outcome 2,

…

$n_m$ of outcome $m$.

This is the multinomial distribution for $\vec{n} = (n_1, \ldots, n_m)$

$$f(\vec{n}; N, \vec{p}) = \frac{N!}{n_1! n_2! \cdots n_m!} p_1^{n_1} p_2^{n_2} \cdots p_m^{n_m}$$

# Multinomial distribution (2)

Now consider outcome *i* as ‘success’, all others as ‘failure’.

→ all $n_i$ individually binomial with parameters $N$, $p_i$

$$E[n_i] = Np_i, \qquad V[n_i] = Np_i(1 - p_i) \quad \text{for all } i$$

One can also find the covariance to be

$$V_{ij} = -Np_i p_j \,, \qquad (i \neq j)$$

Example: $\vec{n} = (n_1, \ldots, n_m)$ represents a histogram with $m$ bins, $N$ total entries, all entries independent.

# Poisson distribution

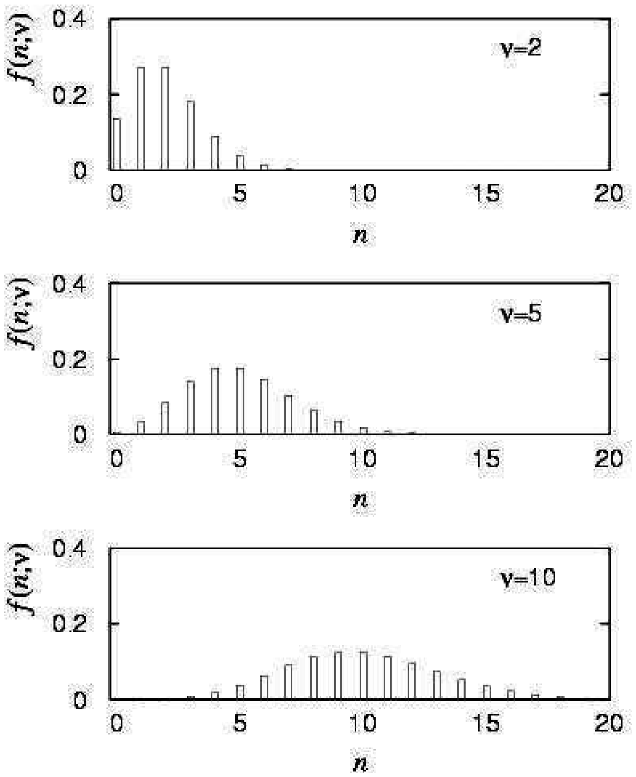
Consider binomial $n$ in the limit

$$N \to \infty, \qquad p \to 0, \qquad E[n] = Np \to \nu \; .$$

$\to$ $n$ follows the Poisson distribution:

$$f(n;\nu) = \frac{\nu^n}{n!} e^{-\nu} \qquad (n \geq 0)$$

$$E[n] = \nu \, , \qquad V[n] = \nu \; .$$




Example: number of scattering events $n$ with cross section $\sigma$ found for a fixed integrated luminosity, with $\nu = \sigma \int L \, dt$ .

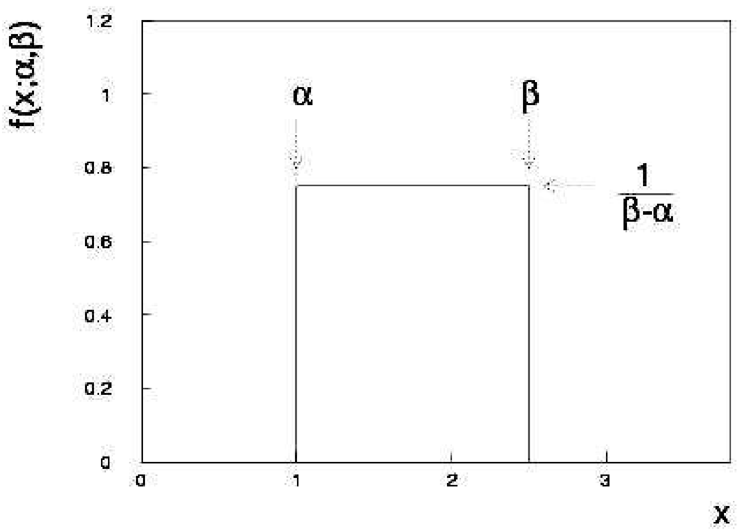
# Uniform distribution

Consider a continuous r.v. $x$ with $-\infty < x < \infty$ . Uniform pdf is:

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta-\alpha} & \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$

$$E[x] = \frac{1}{2}(\alpha+\beta)$$

$$V[x] = \frac{1}{12}(\beta-\alpha)$$




N.B. For any r.v. $x$ with cumulative distribution $F(x)$, $y = F(x)$ is uniform in [0,1].

Example: for $\pi^0 \to \gamma\gamma$, $E_\gamma$ is uniform in $[E_{\min}, E_{\max}]$, with

$$E_{\min} = \frac{1}{2}E_\pi(1-\beta), \qquad E_{\max} = \frac{1}{2}E_\pi(1+\beta)$$

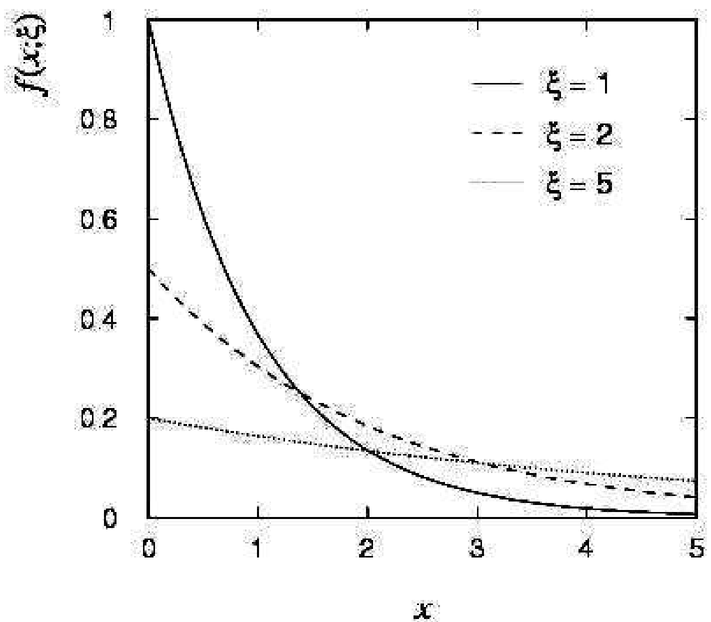
# Exponential distribution

The exponential pdf for the continuous r.v. *x* is defined by:

$$f(x;\xi) = \begin{cases} \frac{1}{\xi} e^{-x/\xi} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[x] = \xi$$

$$V[x] = \xi^2$$




Example: proper decay time *t* of an unstable particle

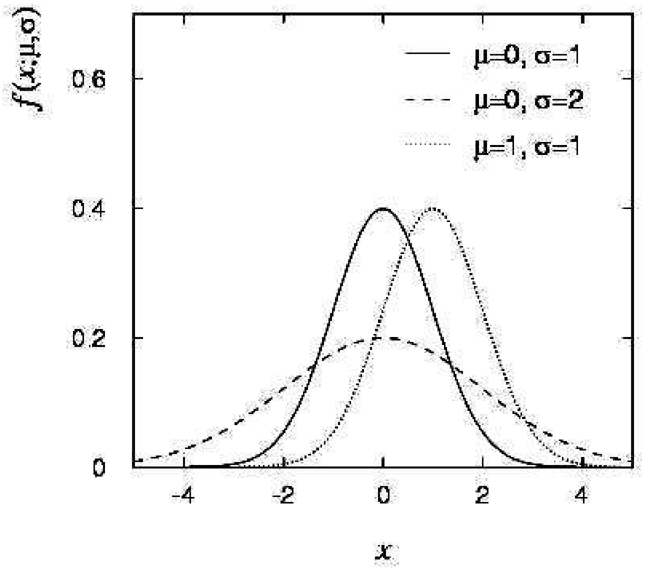
$$f(t;\tau) = \frac{1}{\tau} e^{-t/\tau} \qquad (\tau = \text{mean lifetime})$$

Lack of memory (unique to exponential): $f(t - t_0 | t \geq t_0) = f(t)$

# Gaussian distribution

The Gaussian (normal) pdf for a continuous r.v. $x$ is defined by:

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$




$$E[x] = \mu$$

$$V[x] = \sigma^2$$

(N.B. often $\mu$, $\sigma^2$ denote mean, variance of any r.v., not only Gaussian.)

Special case: $\mu = 0$, $\sigma^2 = 1$ ('standard Gaussian'):

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,, \qquad \Phi(x) = \int_{-\infty}^{x} \varphi(x')\,dx'$$

If $y \sim$ Gaussian with $\mu$, $\sigma^2$, then $x = (y - \mu)/\sigma$ follows $\varphi(x)$.

# Gaussian pdf and the Central Limit Theorem

The Gaussian pdf is so useful because almost any random variable that is a sum of a large number of small contributions follows it. This follows from the Central Limit Theorem:

For $n$ independent r.v.s $x_i$ with finite variances $\sigma_i^2$, otherwise arbitrary pdfs, consider the sum

$$y = \sum_{i=1}^{n} x_i$$

In the limit $n \to \infty$, $y$ is a Gaussian r.v. with

$$E[y] = \sum_{i=1}^{n} \mu_i \qquad V[y] = \sum_{i=1}^{n} \sigma_i^2$$

Measurement errors are often the sum of many contributions, so frequently measured values can be treated as Gaussian r.v.s.

# Central Limit Theorem (2)

The CLT can be proved using characteristic functions (Fourier transforms), see, e.g., SDA Chapter 10.

For finite $n$, the theorem is approximately valid to the extent that the fluctuation of the sum is not dominated by one (or few) terms.

Beware of measurement errors with non-Gaussian tails.

Good example: velocity component $v_x$ of air molecules.

OK example: total deflection due to multiple Coulomb scattering. (Rare large angle deflections give non-Gaussian tail.)

Bad example: energy loss of charged particle traversing thin gas layer. (Rare collisions make up large fraction of energy loss, cf. Landau pdf.)

# Multivariate Gaussian distribution

Multivariate Gaussian pdf for the vector $\vec{x} = (x_1, \ldots, x_n)$ :

$$f(\vec{x}; \vec{\mu}, V) = \frac{1}{(2\pi)^{n/2}|V|^{1/2}} \exp\left[-\frac{1}{2}(\vec{x} - \vec{\mu})^T V^{-1} (\vec{x} - \vec{\mu})\right]$$

$\vec{x}$, $\vec{\mu}$ are column vectors, $\vec{x}^T$, $\vec{\mu}^T$ are transpose (row) vectors,

$$E[x_i] = \mu_i, \; , \qquad \mathrm{cov}[x_i, x_j] = V_{ij} \; .$$

For $n = 2$ this is

$$f(x_1, x_2, ; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

$$\times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]\right\}$$

where $\rho = \mathrm{cov}[x_1, x_2]/(\sigma_1\sigma_2)$ is the correlation coefficient.

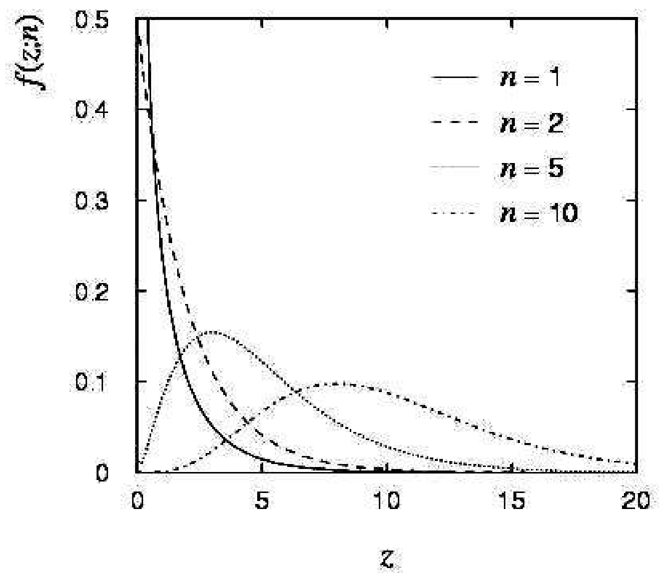
# Chi-square ($\chi^2$) distribution

The chi-square pdf for the continuous r.v. $z$ ($z \geq 0$) is defined by

$$f(z;n) = \frac{1}{2^{n/2}\Gamma(n/2)} z^{n/2-1} e^{-z/2}$$

$n = 1, 2, \ldots =$ number of 'degrees of freedom' (dof)

$$E[z] = n \,, \quad V[z] = 2n \,.$$




For independent Gaussian $x_i$, $i = 1, \ldots, n$, means $\mu_i$, variances $\sigma_i^2$,

$$z = \sum_{i=1}^{n} \frac{(x_i - \mu_i)^2}{\sigma_i^2}$$ follows $\chi^2$ pdf with $n$ dof.

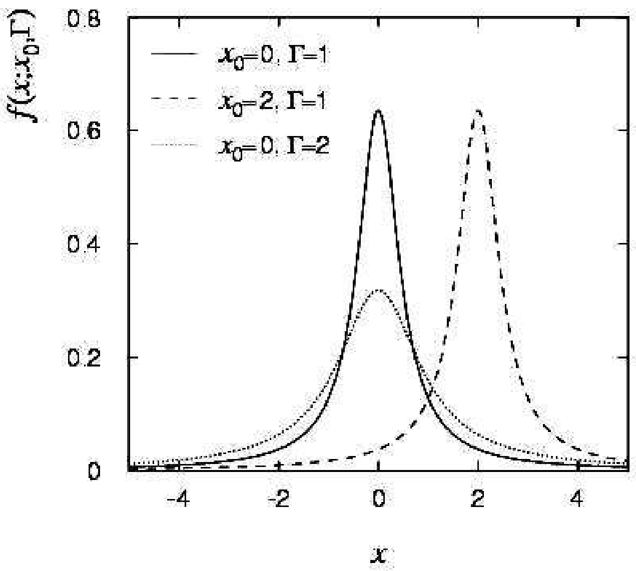
Example: goodness-of-fit test variable especially in conjunction with method of least squares.

# Cauchy (Breit-Wigner) distribution

The Breit-Wigner pdf for the continuous r.v. $x$ is defined by

$$f(x;\Gamma,x_0) = \frac{1}{\pi}\frac{\Gamma/2}{\Gamma^2/4 + (x-x_0)^2}$$

($\Gamma = 2$, $x_0 = 0$ is the Cauchy pdf.)




$E[x]$ not well defined, $V[x] \rightarrow \infty$.

$x_0$ = mode (most probable value)

$\Gamma$ = full width at half maximum

Example: mass of resonance particle, e.g. $\rho$, $K^*$, $\phi^0$, ...

$\Gamma$ = decay rate (inverse of mean lifetime)

# Landau distribution

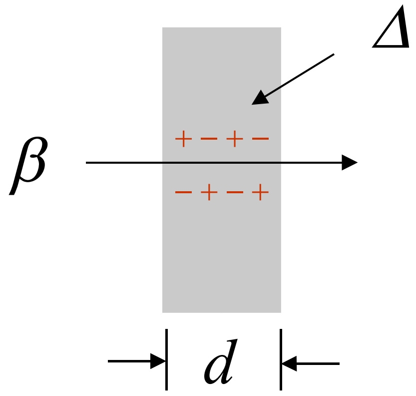
For a charged particle with $\beta = v/c$ traversing a layer of matter of thickness $d$, the energy loss $\Delta$ follows the Landau pdf:

$$f(\Delta;\beta) = \frac{1}{\xi}\phi(\lambda) ,$$

$$\phi(\lambda) = \frac{1}{\pi}\int_0^\infty \exp(-u\ln u - \lambda u)\sin \pi u \, du ,$$

$$\lambda = \frac{1}{\xi}\left[\Delta - \xi\left(\ln\frac{\xi}{\epsilon'} + 1 - \gamma_{\mathrm{E}}\right)\right] ,$$

$$\xi = \frac{2\pi N_{\mathrm{A}} e^4 z^2 \rho \sum Z}{m_{\mathrm{e}} c^2 \sum A}\frac{d}{\beta^2} , \qquad \epsilon' = \frac{I^2 \exp\beta^2}{2 m_{\mathrm{e}} c^2 \beta^2 \gamma^2} .$$




L. Landau, J. Phys. USSR **8** (1944) 201; see also
W. Allison and J. Cobb, Ann. Rev. Nucl. Part. Sci. **30** (1980) 253.

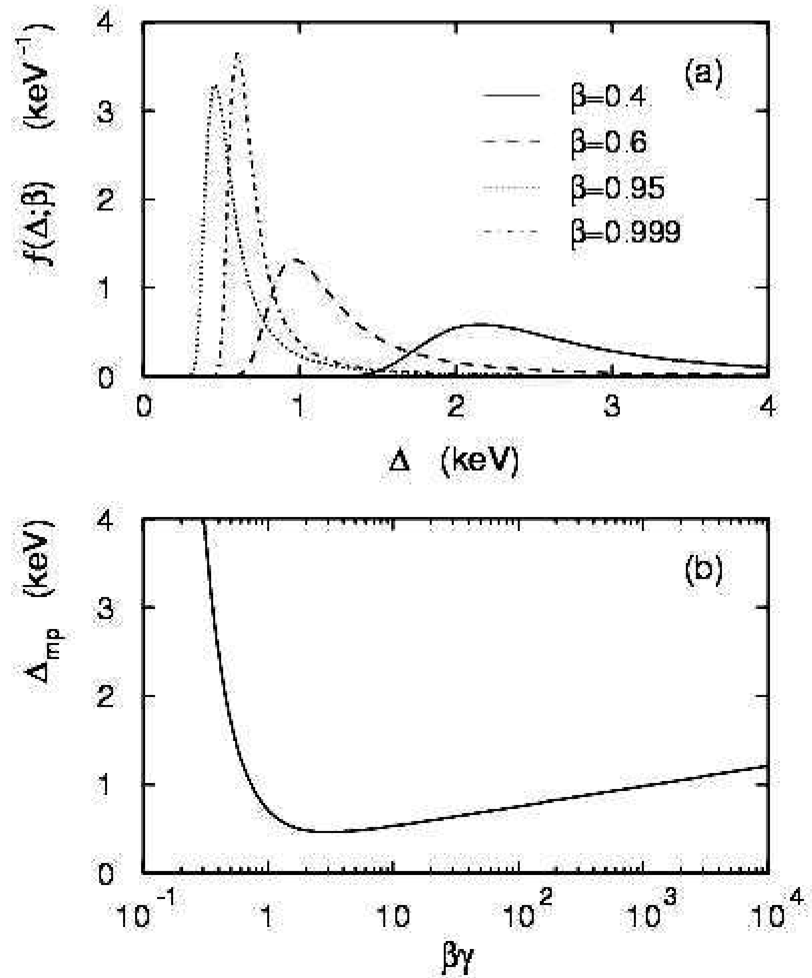
# Landau distribution (2)

Long ‘Landau tail’

→ all moments ∞

Mode (most probable value) sensitive to $\beta$,

→ particle i.d.



# Wrapping up lecture 1

Up to now we've talked only about probability:

definition and interpretation,
Bayes' theorem,
random variables,
probability density functions,
expectation values,
catalogue of distributions, ...

But suppose now we are faced with experimental data.
We want to infer something about the (probabilistic) processes that produced the data.

This is statistics, the main subject of the next two lectures.