Statistical Methods for Particle Physics (I)

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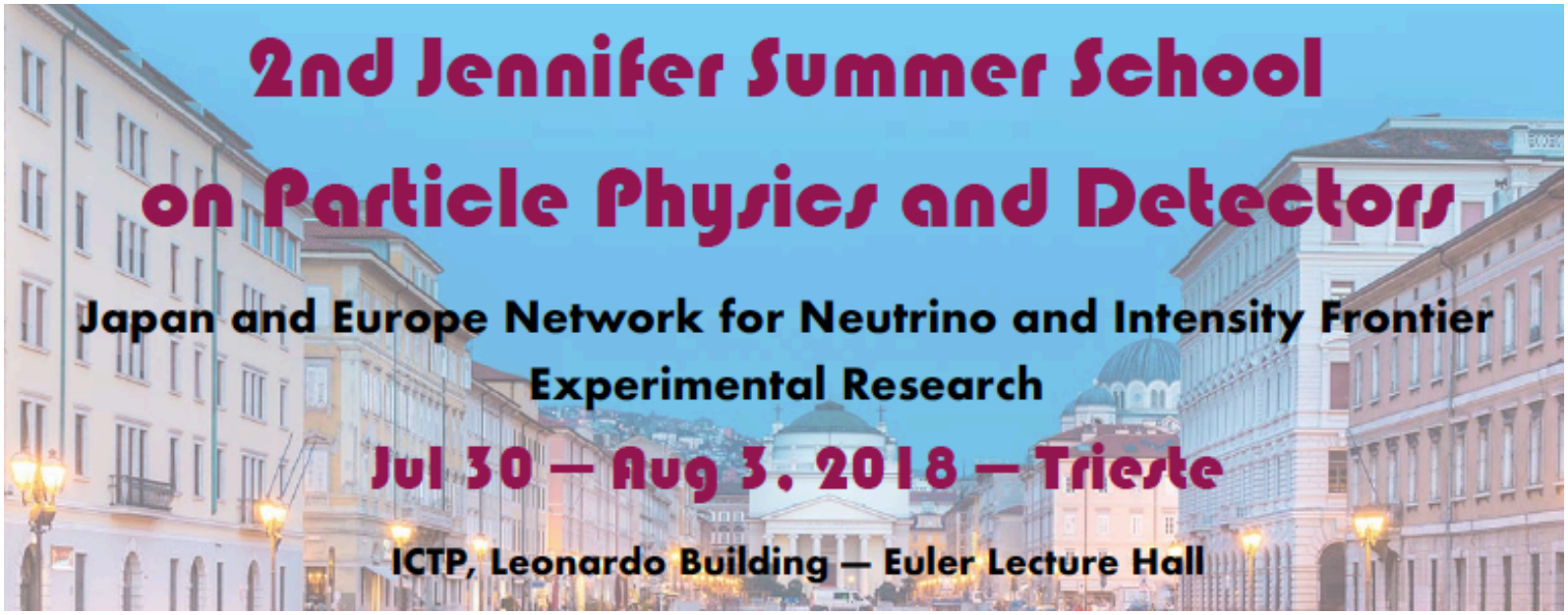

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Outline

→ Lecture 1:

Quick review of probability
Parameter estimation, maximum likelihood
Statistical tests for discovery and limits

Lecture 2:

Limits for the Poisson counting experiment
Nuisance parameters and systematic uncertainties
Tests from profile likelihood ratio
More parameter estimation, Bayesian methods
(Experimental sensitivity)

Some statistics books, papers, etc.

G. Cowan, *Statistical Data Analysis*, Clarendon, Oxford, 1998

R.J. Barlow, *Statistics: A Guide to the Use of Statistical Methods in the Physical Sciences*, Wiley, 1989

Ilya Narsky and Frank C. Porter, *Statistical Analysis Techniques in Particle Physics*, Wiley, 2014.

L. Lyons, *Statistics for Nuclear and Particle Physics*, CUP, 1986

F. James., *Statistical and Computational Methods in Experimental Physics*, 2nd ed., World Scientific, 2006

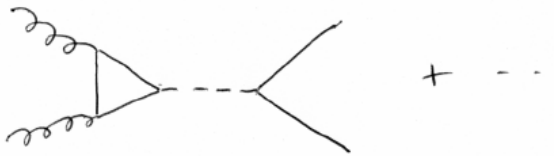
S. Brandt, *Statistical and Computational Methods in Data Analysis*, Springer, New York, 1998 (with program library on CD)

C. Patrignani et al. (Particle Data Group), *Review of Particle Physics*, Chin. Phys. C, 40, 100001 (2016); see also **`pdg.lbl.gov`** sections on probability, statistics, Monte Carlo

Theory ↔ Statistics ↔ Experiment

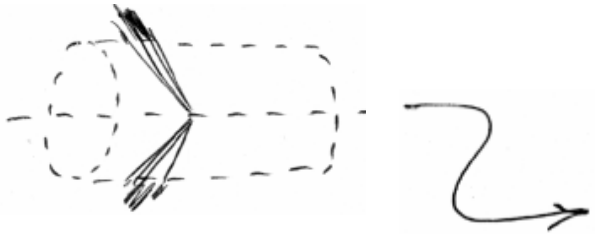
Theory (model, hypothesis):

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \not{D} \psi + \cdots$$

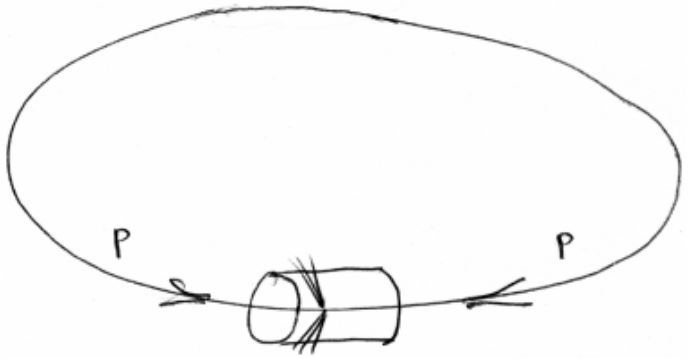


$$\sigma = \frac{G_F \alpha_s^2 m_H^2}{288 \sqrt{2\pi}} \times$$

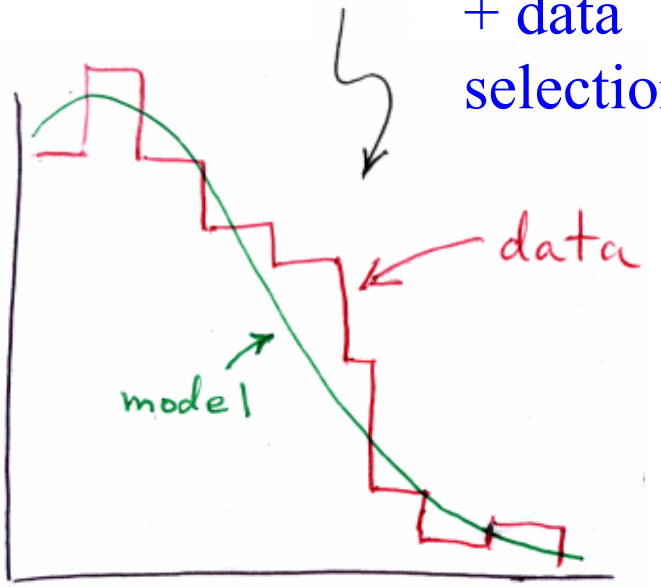
+ simulation of detector and cuts



Experiment:



+ data selection



Data analysis in particle physics

Observe events (e.g., pp collisions) and for each, measure a set of characteristics:

particle momenta, number of muons, energy of jets,...

Compare observed distributions of these characteristics to predictions of theory. From this, we want to:

Estimate the free parameters of the theory: $m_H = 125.4$

Quantify the uncertainty in the estimates: $\pm$ 0.4 GeV

Assess how well a given theory stands in agreement with the observed data: 0^+ good, 2^+ bad

To do this we need a clear definition of PROBABILITY

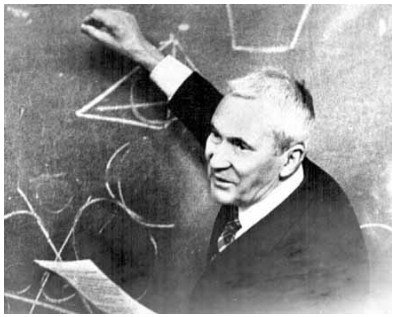
A definition of probability

Consider a set S with subsets A, B, ...

For all $A \subset S, P(A) \geq 0$

$$P(S) = 1$$

If $A \cap B = \emptyset, P(A \cup B) = P(A) + P(B)$



Kolmogorov axioms (1933)

Also define conditional probability of A given B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Subsets A, B independent if: $P(A \cap B) = P(A)P(B)$

If A, B independent, $P(A|B) = \dfrac{P(A)P(B)}{P(B)} = P(A)$

Interpretation of probability

I. Relative frequency

A, B, ... are outcomes of a repeatable experiment

$$P(A) = \lim_{n\to\infty} \frac{\text{times outcome is } A}{n}$$

cf. quantum mechanics, particle scattering, radioactive decay...

II. Subjective probability

A, B, ... are hypotheses (statements that are true or false)

$$P(A) = \text{degree of belief that } A \text{ is true}$$

- Both interpretations consistent with Kolmogorov axioms.
- In particle physics frequency interpretation often most useful, but subjective probability can provide more natural treatment of non-repeatable phenomena:

systematic uncertainties, probability that Higgs boson exists,...

Bayes' theorem

From the definition of conditional probability we have,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B|A) = \frac{P(B \cap A)}{P(A)}$$

but $P(A \cap B) = P(B \cap A)$, so

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Bayes' theorem

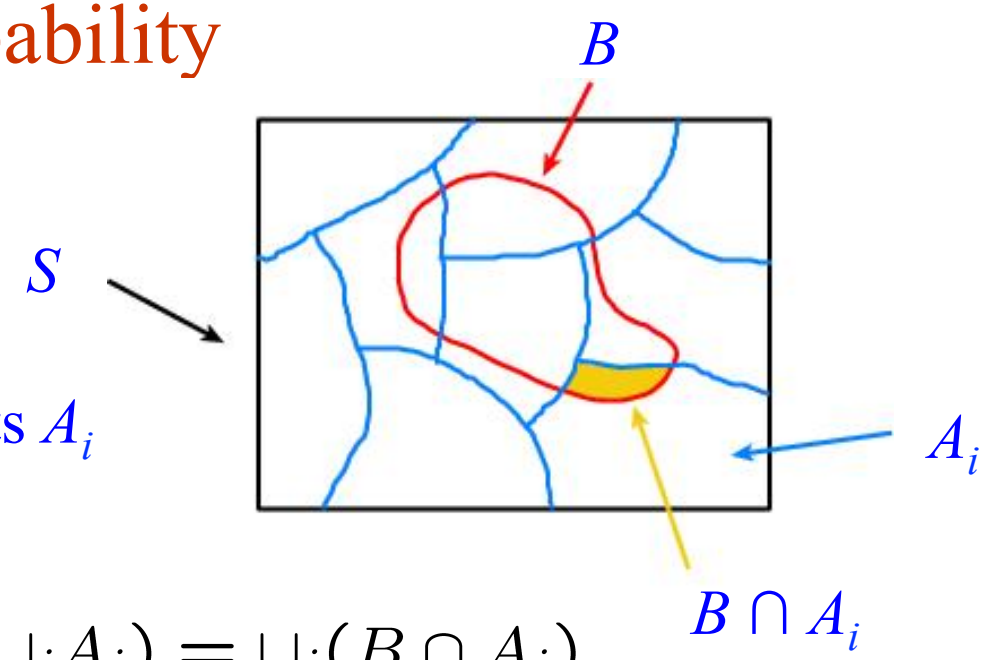
First published (posthumously) by the Reverend Thomas Bayes (1702−1761)

An essay towards solving a problem in the doctrine of chances, Philos. Trans. R. Soc. **53** (1763) 370; reprinted in Biometrika, **45** (1958) 293.

The law of total probability

Consider a subset B of the sample space S,

divided into disjoint subsets A_i such that $\cup_i A_i = S$,



$$\rightarrow \quad B = B \cap S = B \cap (\cup_i A_i) = \cup_i (B \cap A_i),$$

$$\rightarrow \quad P(B) = P(\cup_i (B \cap A_i)) = \sum_i P(B \cap A_i)$$

$$\rightarrow \quad P(B) = \sum_i P(B|A_i) P(A_i)$$ law of total probability

Bayes' theorem becomes

$$P(A|B) = \frac{P(B|A)P(A)}{\sum_i P(B|A_i)P(A_i)}$$

An example using Bayes' theorem

Suppose the probability (for anyone) to have a disease D is:

$$P(\mathrm{D}) = 0.001$$
$$P(\text{no D}) = 0.999$$

← prior probabilities, i.e., before any test carried out

Consider a test for the disease: result is + or −

$$P(+|\mathrm{D}) = 0.98$$
$$P(-|\mathrm{D}) = 0.02$$

← probabilities to (in)correctly identify a person with the disease

$$P(+|\text{no D}) = 0.03$$
$$P(-|\text{no D}) = 0.97$$

← probabilities to (in)correctly identify a healthy person

Suppose your result is +. How worried should you be?

Bayes' theorem example (cont.)

The probability to have the disease given a + result is

$$p(\mathrm{D}|+) = \frac{P(+|\mathrm{D})P(\mathrm{D})}{P(+|\mathrm{D})P(\mathrm{D}) + P(+|\text{no D})P(\text{no D})}$$

$$= \frac{0.98 \times 0.001}{0.98 \times 0.001 + 0.03 \times 0.999}$$

$$= 0.032 \quad \leftarrow \text{posterior probability}$$

i.e. you're probably OK!

Your viewpoint: my degree of belief that I have the disease is 3.2%.

Your doctor's viewpoint: 3.2% of people like this have the disease.

Frequentist Statistics − general philosophy

In frequentist statistics, probabilities are associated only with the data, i.e., outcomes of repeatable observations (shorthand: $\vec{x}$).

Probability = limiting frequency

Probabilities such as

P (Higgs boson exists),
P ($0.117 < \alpha_s < 0.121$),

etc. are either 0 or 1, but we don't know which.

The tools of frequentist statistics tell us what to expect, under the assumption of certain probabilities, about hypothetical repeated observations.

A hypothesis is is preferred if the data are found in a region of high predicted probability (i.e., where an alternative hypothesis predicts lower probability).

Bayesian Statistics – general philosophy

In Bayesian statistics, use subjective probability for hypotheses:

probability of the data assuming hypothesis H (the likelihood)

prior probability, i.e., before seeing the data

$$P(H|\vec{x}) = \frac{P(\vec{x}|H)\pi(H)}{\int P(\vec{x}|H)\pi(H)\,dH}$$

posterior probability, i.e., after seeing the data

normalization involves sum over all possible hypotheses

Bayes' theorem has an "if-then" character: If your prior probabilities were $\pi(H)$, then it says how these probabilities should change in the light of the data.

No general prescription for priors (subjective!)

The likelihood function

Suppose the entire result of an experiment (set of measurements) is a collection of numbers $\boldsymbol{x}$, and suppose the joint pdf for the data $\boldsymbol{x}$ is a function that depends on a set of parameters $\boldsymbol{\theta}$:

$$P(\mathbf{x}|\boldsymbol{\theta})$$

Now evaluate this function with the data obtained and regard it as a function of the parameter(s). This is the likelihood function:

$$L(\boldsymbol{\theta}) = P(\mathbf{x}|\boldsymbol{\theta})$$

($\boldsymbol{x}$ constant)

The likelihood function for i.i.d.*. data

* i.i.d. = independent and identically distributed

Consider n independent observations of x: $x_1, \ldots, x_n$, where x follows $f(x; \theta)$. The joint pdf for the whole data sample is:

$$f(x_1, \ldots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$$

In this case the likelihood function is

$$L(\vec{\theta}) = \prod_{i=1}^{n} f(x_i; \vec{\theta}) \qquad (x_i \text{ constant})$$

Frequentist parameter estimation

Suppose we have a pdf characterized by one or more parameters:

$$f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$$

random variable parameter

Suppose we have a sample of observed values: $\vec{x} = (x_1, \ldots, x_n)$

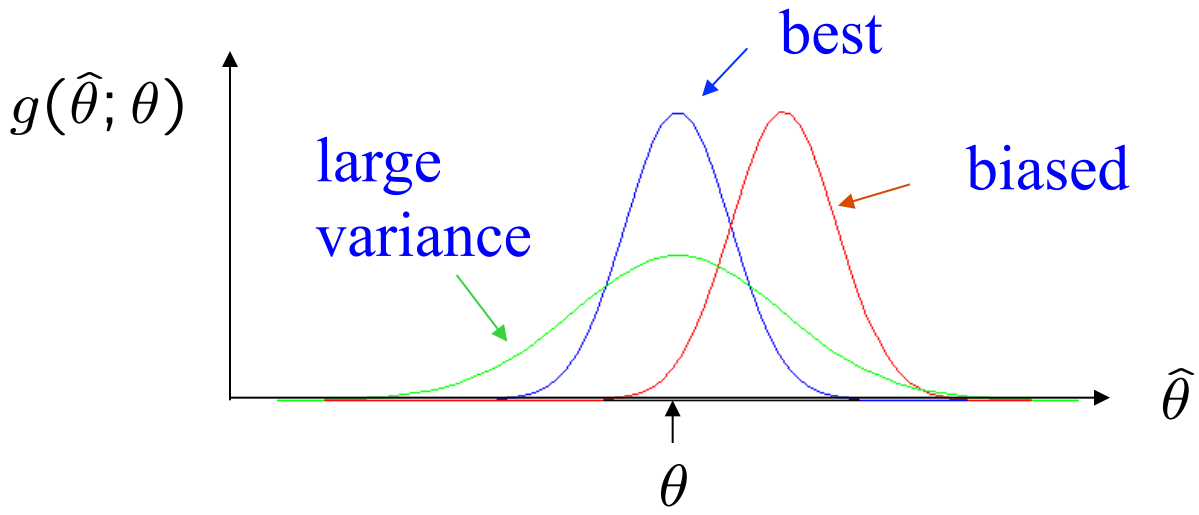
We want to find some function of the data to estimate the parameter(s):

$\widehat{\theta}(\vec{x})$ ← estimator written with a hat

Sometimes we say 'estimator' for the function of x_1, ..., x_n;
'estimate' for the value of the estimator with a particular data set.

Properties of estimators

Estimators are functions of the data and thus characterized by a sampling distribution with a given (co)variance:

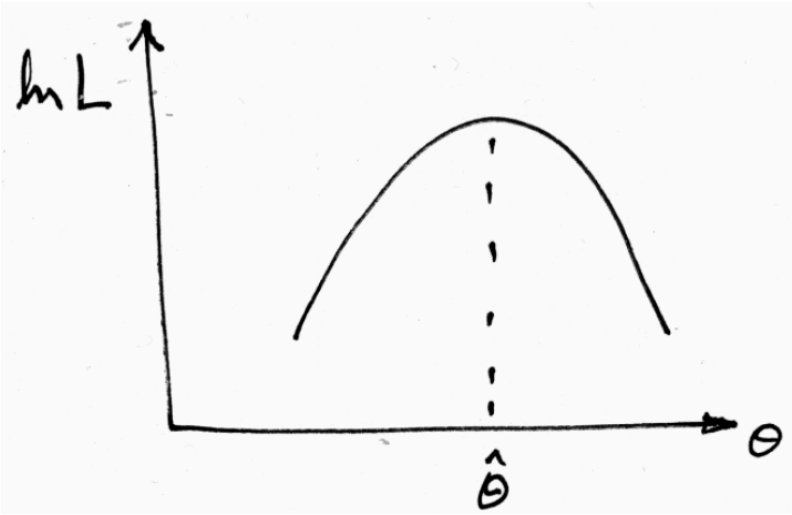


In general they may have a nonzero bias: $b = E[\hat{\theta}] - \theta$

Want small variance and small bias, but in general cannot optimize with respect to both; some trade-off necessary.

Maximum Likelihood (ML) estimators

The most important frequentist method for constructing estimators is to take the value of the parameter(s) that maximize the likelihood (or equivalently the log-likelihod):



$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}}\, L(x|\theta)$$

In some cases we can find the ML estimator as a closed-form function of the data; more often it is found numerically.

ML example: parameter of exponential pdf

Consider exponential pdf, $f(t;\tau) = \frac{1}{\tau} e^{-t/\tau}$

and suppose we have i.i.d. data, $t_1, \ldots, t_n$

The likelihood function is $L(\tau) = \prod_{i=1}^{n} \frac{1}{\tau} e^{-t_i/\tau}$

The value of τ for which $L(\tau)$ is maximum also gives the maximum value of its logarithm (the log-likelihood function):

$$\ln L(\tau) = \sum_{i=1}^{n} \ln f(t_i;\tau) = \sum_{i=1}^{n} \left(\ln \frac{1}{\tau} - \frac{t_i}{\tau} \right)$$

ML example: parameter of exponential pdf (2)

Find its maximum by setting $\dfrac{\partial \ln L(\tau)}{\partial \tau} = 0$,

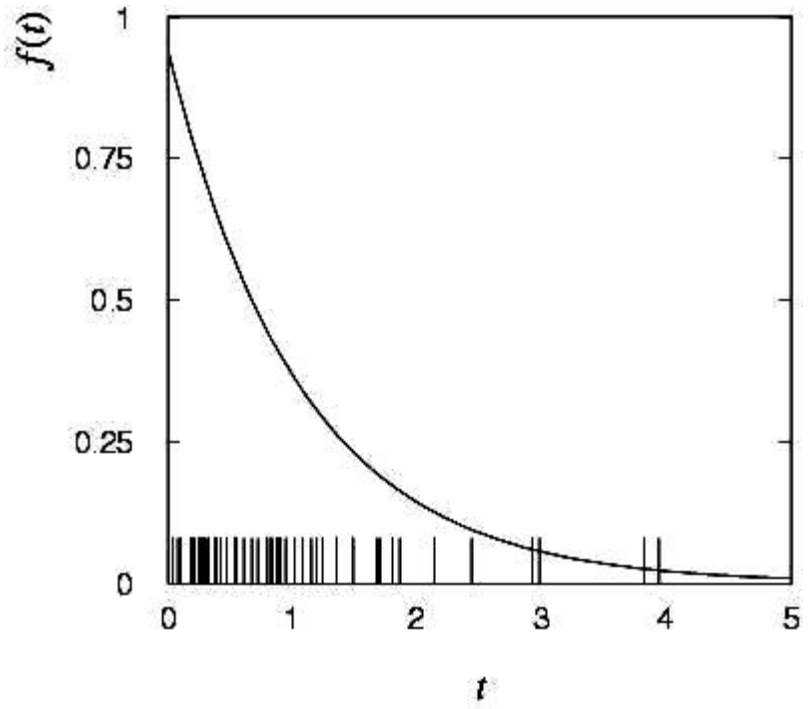
$$\rightarrow \quad \hat{\tau} = \frac{1}{n}\sum_{i=1}^{n} t_i$$

Monte Carlo test:

generate 50 values using $\tau = 1$:

We find the ML estimate:

$$\hat{\tau} = 1.062$$



ML example: parameter of exponential pdf (3)

For the exponential distribution one has for mean, variance:

$$E[t] = \int_0^\infty t \frac{1}{\tau} e^{-t/\tau}\, dt = \tau$$

$$V[t] = \int_0^\infty (t-\tau)^2 \frac{1}{\tau} e^{-t/\tau}\, dt = \tau^2$$

For the ML estimator $\hat{\tau} = \frac{1}{n}\sum_{i=1}^{n} t_i$ we therefore find

$$E[\hat{\tau}] = E\left[\frac{1}{n}\sum_{i=1}^{n} t_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[t_i] = \tau \quad \longrightarrow \quad b = E[\hat{\tau}] - \tau = 0$$

$$V[\hat{\tau}] = V\left[\frac{1}{n}\sum_{i=1}^{n} t_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} V[t_i] = \frac{\tau^2}{n} \quad \longrightarrow \quad \sigma_{\hat{\tau}} = \frac{\tau}{\sqrt{n}}$$

Variance of estimators: Monte Carlo method

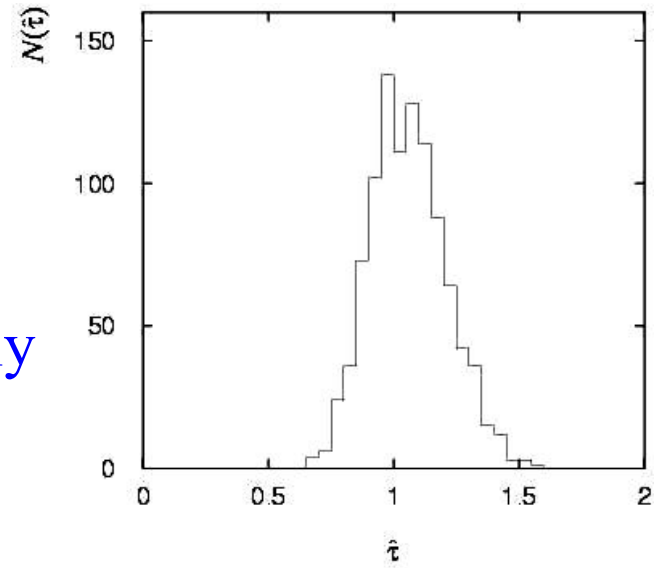
Having estimated our parameter we now need to report its 'statistical error', i.e., how widely distributed would estimates be if we were to repeat the entire measurement many times.

One way to do this would be to simulate the entire experiment many times with a Monte Carlo program (use ML estimate for MC).

For exponential example, from sample variance of estimates we find:

$$\hat{\sigma}_{\hat{\tau}} = 0.151$$

Note distribution of estimates is roughly Gaussian − (almost) always true for ML in large sample limit.



Variance of estimators from information inequality

The information inequality (RCF) sets a lower bound on the variance of any estimator (not only ML):

$$V[\hat{\theta}] \geq \left(1 + \frac{\partial b}{\partial \theta}\right)^2 \Big/ E\left[-\frac{\partial^2 \ln L}{\partial \theta^2}\right]$$

Minimum Variance Bound (MVB)

$$(b = E[\hat{\theta}] - \theta)$$

Often the bias b is small, and equality either holds exactly or is a good approximation (e.g. large data sample limit). Then,

$$V[\hat{\theta}] \approx -1 \Big/ E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right]$$

Estimate this using the 2nd derivative of $\ln L$ at its maximum:

$$\hat{V}[\hat{\theta}] = -\left(\frac{\partial^2 \ln L}{\partial \theta^2}\right)^{-1}\Bigg|_{\theta=\hat{\theta}}$$

Variance of estimators: graphical method

Expand $\ln L(\theta)$ about its maximum:

$$\ln L(\theta) = \ln L(\hat{\theta}) + \left[\frac{\partial \ln L}{\partial \theta}\right]_{\theta=\hat{\theta}} (\theta - \hat{\theta}) + \frac{1}{2!}\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right]_{\theta=\hat{\theta}} (\theta - \hat{\theta})^2 + \ldots$$

First term is $\ln L_{\max}$, second term is zero, for third term use information inequality (assume equality):

$$\ln L(\theta) \approx \ln L_{\max} - \frac{(\theta - \hat{\theta})^2}{2\widehat{\sigma^2}_{\hat{\theta}}}$$

$$\text{i.e.,} \qquad \ln L(\hat{\theta} \pm \hat{\sigma}_{\hat{\theta}}) \approx \ln L_{\max} - \frac{1}{2}$$

$\rightarrow$ to get $\hat{\sigma}_{\hat{\theta}}$, change θ away from $\hat{\theta}$ until $\ln L$ decreases by 1/2.

Example of variance by graphical method

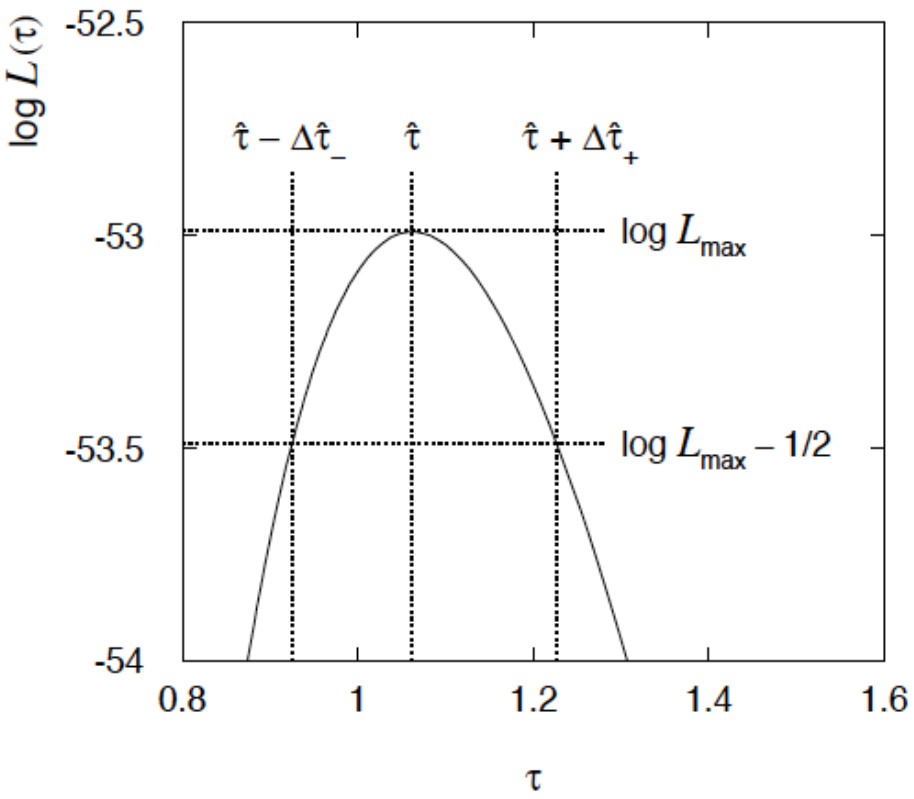
ML example with exponential:

$$\hat{\tau} = 1.062$$

$$\Delta\hat{\tau}_- = 0.137$$

$$\Delta\hat{\tau}_+ = 0.165$$

$$\hat{\sigma}_{\hat{\tau}} \approx \Delta\hat{\tau}_- \approx \Delta\hat{\tau}_+ \approx 0.15$$



Not quite parabolic ln L since finite sample size (n = 50).

Information inequality for N parameters

Suppose we have estimated N parameters $\vec{\theta} = (\theta_1, \ldots, \theta_N)$.

The (inverse) minimum variance bound is given by the Fisher information matrix:

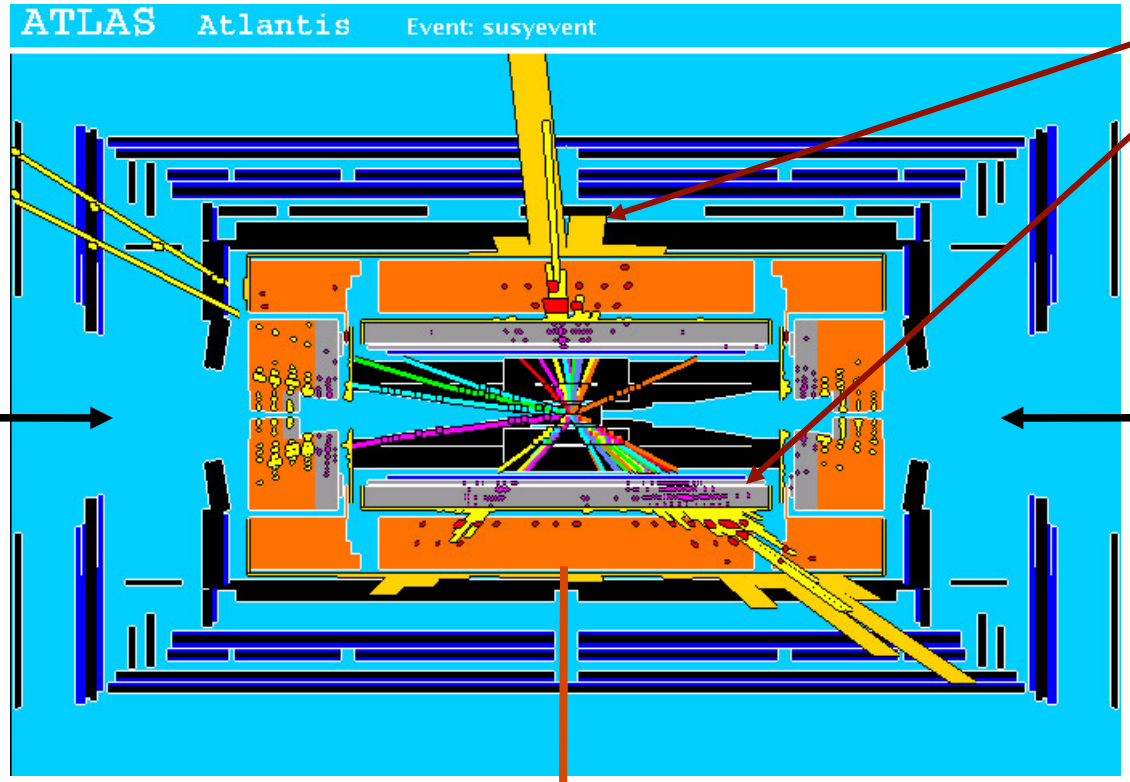
$$I_{ij} = E\left[-\frac{\partial^2 \ln L}{\partial\theta_i \partial\theta_j}\right] = -n\int f(x;\vec{\theta})\frac{\partial^2 \ln f(x;\vec{\theta})}{\partial\theta_i \partial\theta_j}\,dx$$

The information inequality then states that $V - I^{-1}$ is a positive semi-definite matrix, where $V_{ij} = \mathrm{cov}[\hat{\theta}_i, \hat{\theta}_j]$. Therefore

$$V[\hat{\theta}_i] \geq (I^{-1})_{ii}$$

Often use I^{-1} as an approximation for covariance matrix, estimate using e.g. matrix of 2nd derivatives at maximum of L.

Prelude to statistical tests: A simulated SUSY event



high p_T muons

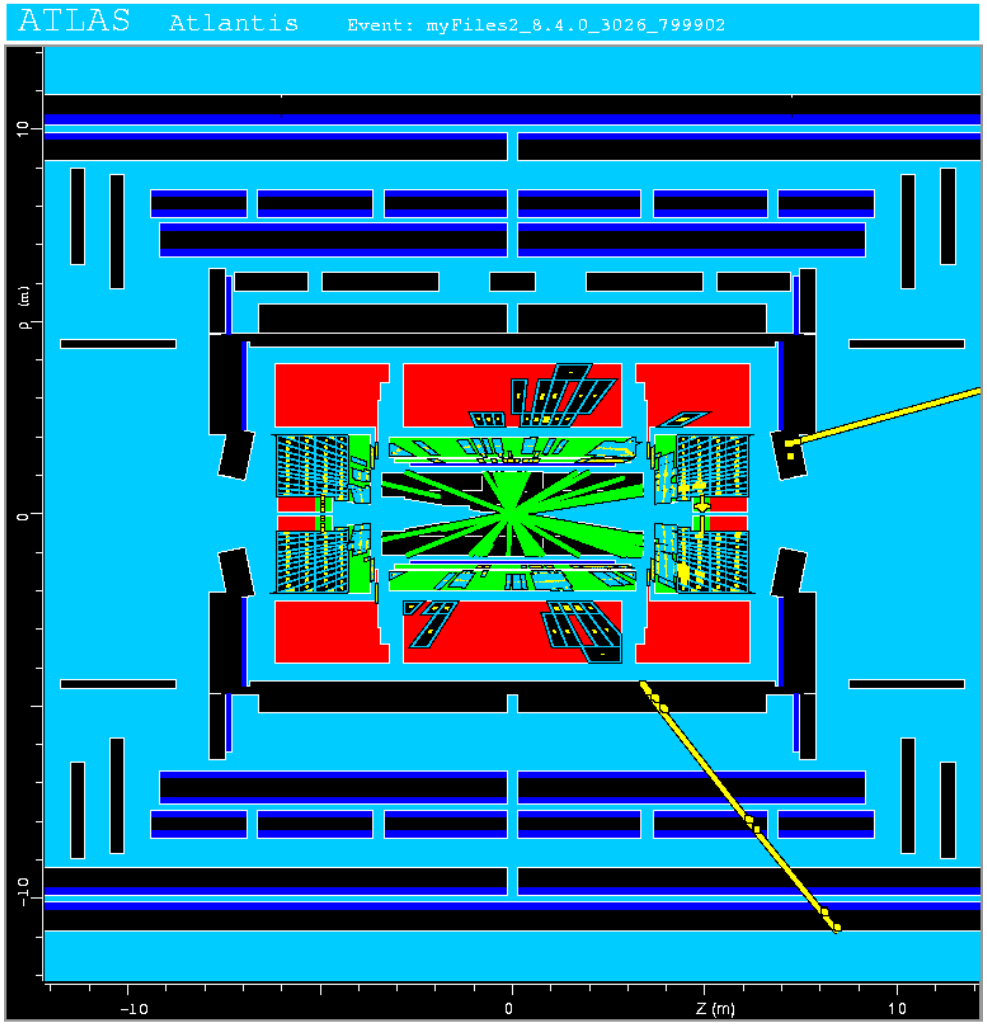
high p_T jets of hadrons

p

p

missing transverse energy

Background events



This event from Standard Model ttbar production also has high p_T jets and muons, and some missing transverse energy.

→ can easily mimic a SUSY event.

Frequentist statistical tests

Suppose a measurement produces data $\boldsymbol{x}$; consider a hypothesis H_0 we want to test and alternative H_1

$$H_0, H_1 \text{ specify probability for } \boldsymbol{x}\text{: } P(\boldsymbol{x}|H_0), P(\boldsymbol{x}|H_1)$$

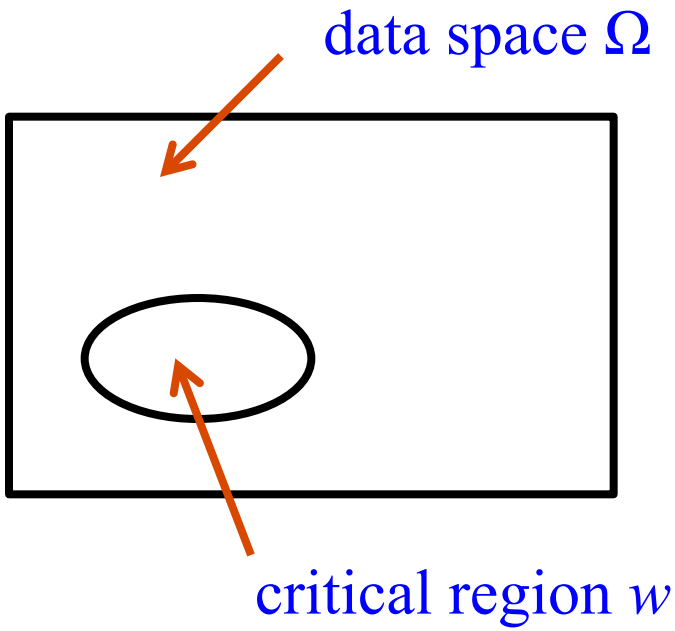
A test of H_0 is defined by specifying a critical region w of the data space such that there is no more than some (small) probability α, assuming H_0 is correct, to observe the data there, i.e.,

$$P(x \in w \,|\, H_0) \leq \alpha$$

Need inequality if data are discrete.

α is called the size or significance level of the test.

If x is observed in the critical region, reject H_0.

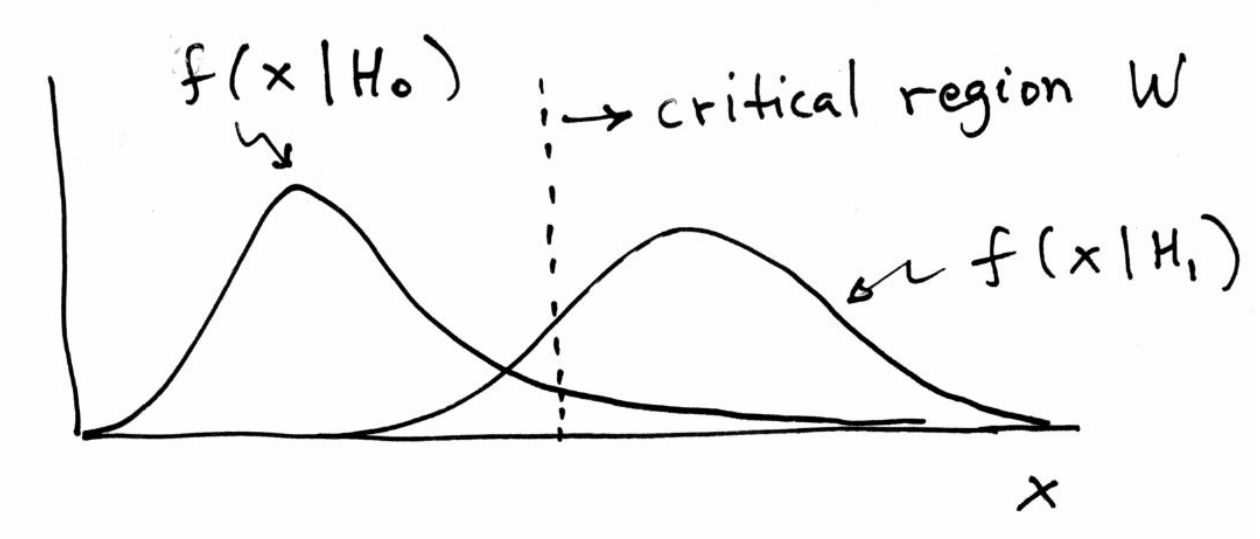


Definition of a test (2)

But in general there are an infinite number of possible critical regions that give the same significance level α.

So the choice of the critical region for a test of H_0 needs to take into account the alternative hypothesis H_1.

Roughly speaking, place the critical region where there is a low probability to be found if H_0 is true, but high if H_1 is true:



Classification viewed as a statistical test

Probability to reject H_0 if true (type I error): $\alpha = \int_W f(\mathbf{x}|H_0)d\mathbf{x}$

α = size of test, significance level, false discovery rate

Probability to accept H_0 if H_1 true (type II error) $\beta = \int_{\overline{W}} f(\mathbf{x}|H_1)d\mathbf{x}$

$1 - \beta$ = power of test with respect to H_1

Equivalently if e.g. H_0 = background, H_1 = signal, use efficiencies:

$$\varepsilon_\mathrm{b} = \int_W f(\mathbf{x}|H_0) = \alpha$$

$$\varepsilon_\mathrm{s} = \int_W f(\mathbf{x}|H_1) = 1 - \beta = \text{power}$$

Purity / misclassification rate

Consider the probability that an event of signal (s) type classified correctly (i.e., the event selection purity),

Use Bayes' theorem:

Here W is signal region

ε_s

prior probability

$$P(\mathrm{s}|\mathbf{x} \in W) = \frac{P(\mathbf{x} \in W|\mathrm{s})P(\mathrm{s})}{P(\mathbf{x} \in W|\mathrm{s})P(\mathrm{s}) + P(\mathbf{x} \in W|\mathrm{b})P(\mathrm{b})}$$

ε_b

posterior probability = signal purity

= 1 – signal misclassification rate

Note purity depends on the prior probability for an event to be signal or background as well as on s/b efficiencies.

Physics context of a statistical test

Event Selection: data = individual event; goal is to classify

Example: separation of different particle types (electron vs muon) or known event types (ttbar vs QCD multijet).
E.g. test H_0 : event is background vs. H_1 : event is signal.
Use selected events for further study.

Search for New Physics: data = a sample of events. Test null hypothesis

H_0 : all events correspond to Standard Model (background only),

against the alternative

H_1 : events include a type whose existence is not yet established (signal plus background)

Many subtle issues here, mainly related to the high standard of proof required to establish presence of a new phenomenon. The optimal statistical test for a search is closely related to that used for event selection.

Statistical tests for event selection

Suppose the result of a measurement for an individual event is a collection of numbers $\vec{x} = (x_1, \ldots, x_n)$

x_1 = number of muons,

x_2 = mean p_T of jets,

x_3 = missing energy, ...

$\vec{x}$ follows some n-dimensional joint pdf, which depends on the type of event produced, i.e., was it

$$pp \to t\bar{t} \ , \quad pp \to \tilde{g}\tilde{g} \ , \ldots$$

For each reaction we consider we will have a hypothesis for the pdf of $\boldsymbol{x}$, e.g., $p(\boldsymbol{x}|\mathrm{b})$, $p(\boldsymbol{x}|\mathrm{s})$

E.g. here call H_0 the background hypothesis (the event type we want to reject); H_1 is signal hypothesis (the type we want).

Selecting events

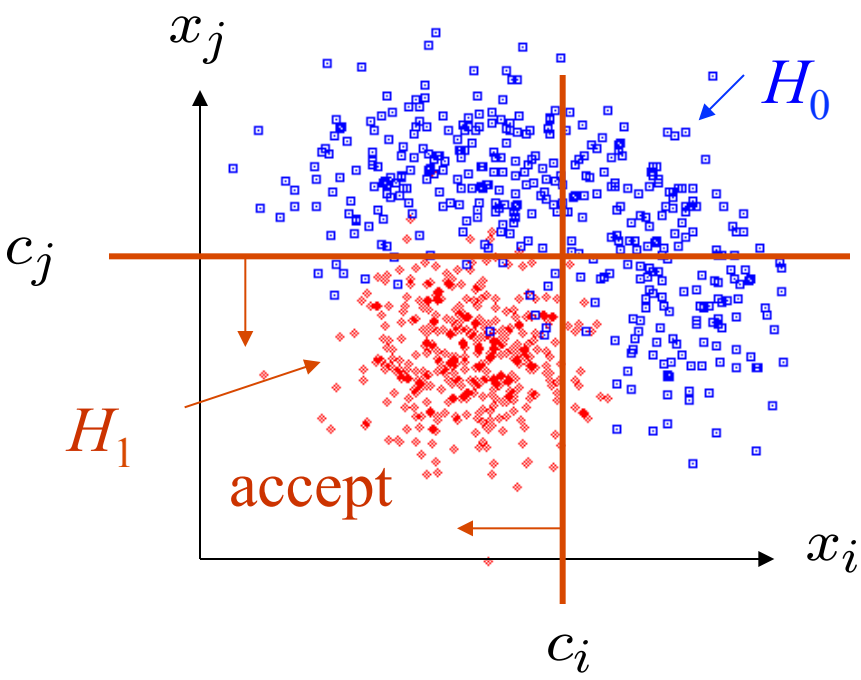
Suppose we have a data sample with two kinds of events, corresponding to hypotheses H_0 and H_1 and we want to select those of type H_1.

Each event is a point in $\vec{x}$ space. What 'decision boundary' should we use to accept/reject events as belonging to event types H_0 or H_1?

Perhaps select events with 'cuts':

$$x_i < c_i$$

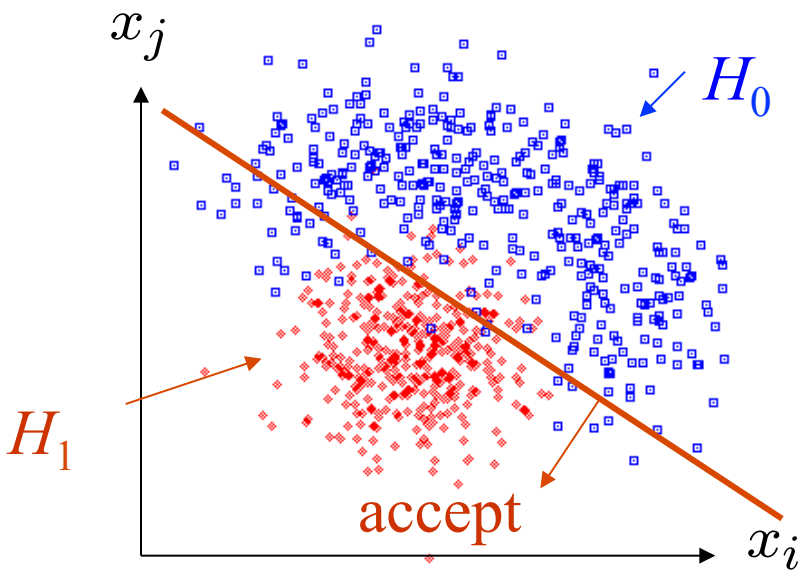
$$x_j < c_j$$



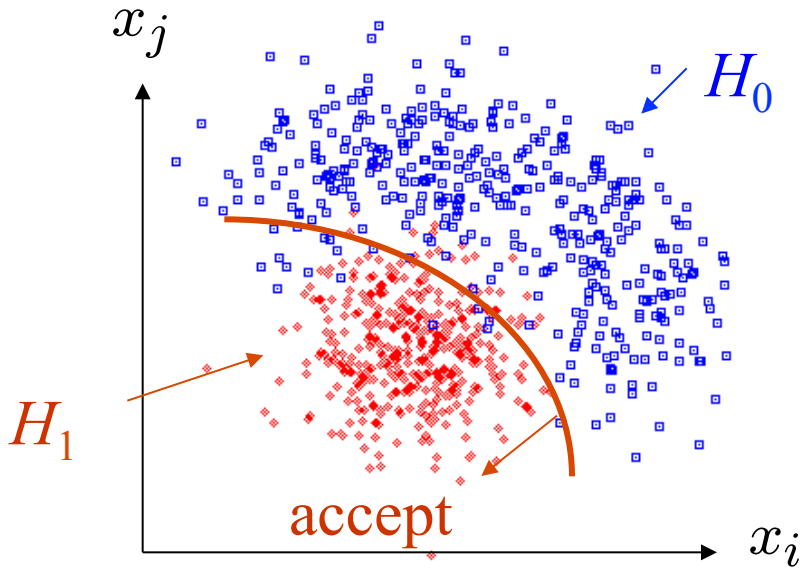
Other ways to select events

Or maybe use some other sort of decision boundary:

linear



or nonlinear



How can we do this in an 'optimal' way?

Test statistics

The boundary of the critical region for an n-dimensional data space $\boldsymbol{x} = (x_1,..., x_n)$ can be defined by an equation of the form

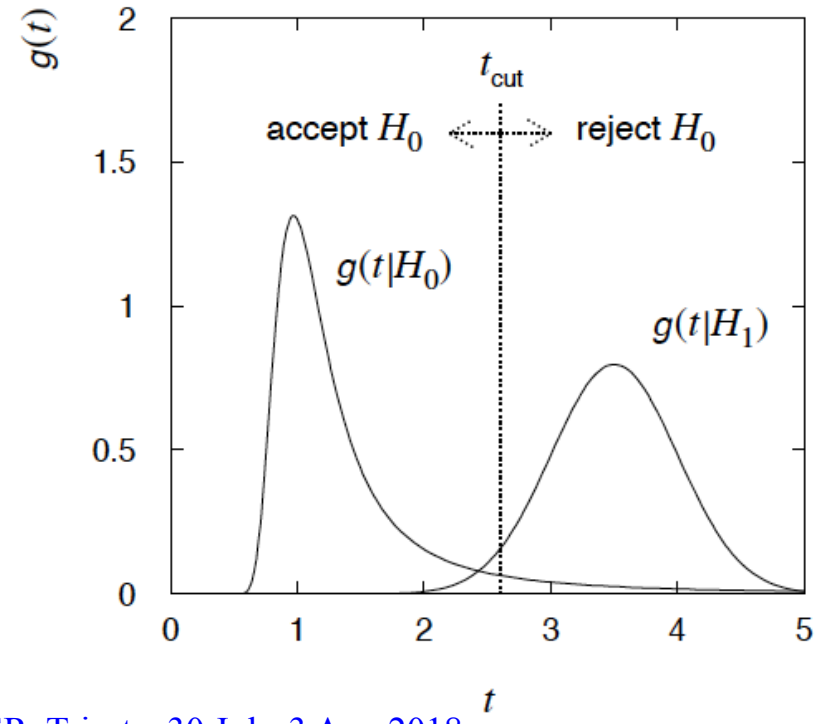
$$t(x_1, \ldots, x_n) = t_{\text{cut}}$$

where $t(x_1,\ldots, x_n)$ is a scalar test statistic.

We can work out the pdfs $g(t|H_0),\ g(t|H_1),\ \ldots$

Decision boundary is now a single 'cut' on t, defining the critical region.

So for an n-dimensional problem we have a corresponding 1-d problem.



Test statistic based on likelihood ratio

How can we choose a test's critical region in an 'optimal way'?

Neyman-Pearson lemma states:

To get the highest power for a given significance level in a test of H_0, (background) versus H_1, (signal) the critical region should have

$$\frac{f(\mathbf{x}|H_1)}{f(\mathbf{x}|H_0)} > c$$

inside the region, and $\leq c$ outside, where c is a constant chosen to give a test of the desired size.

Equivalently, optimal scalar test statistic is

$$t(\mathbf{x}) = \frac{f(\mathbf{x}|H_1)}{f(\mathbf{x}|H_0)}$$

N.B. any monotonic function of this is leads to the same test.

Neyman-Pearson doesn't usually help

We usually don't have explicit formulae for the pdfs $f(\boldsymbol{x}|\mathrm{s})$, $f(\boldsymbol{x}|\mathrm{b})$, so for a given $\boldsymbol{x}$ we can't evaluate the likelihood ratio

$$t(\mathbf{x}) = \frac{f(\mathbf{x}|s)}{f(\mathbf{x}|b)}$$

Instead we may have Monte Carlo models for signal and background processes, so we can produce simulated data:

$$\text{generate } \boldsymbol{x} \sim f(\boldsymbol{x}|\mathrm{s}) \quad \rightarrow \quad \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$$

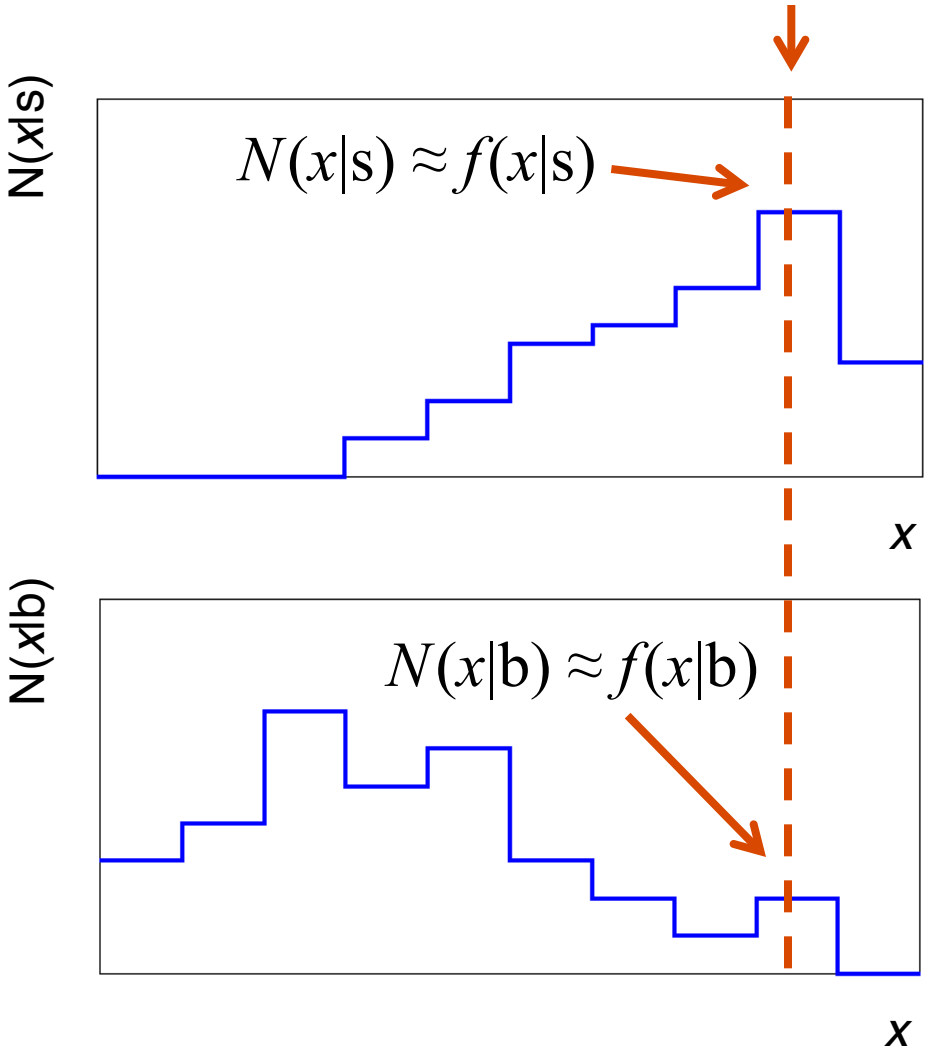
$$\text{generate } \boldsymbol{x} \sim f(\boldsymbol{x}|\mathrm{b}) \quad \rightarrow \quad \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$$

This gives samples of "training data" with events of known type.

Can be expensive (1 fully simulated LHC event ~ 1 CPU minute).

Approximate LR from histograms

Want $t(x) = f(x|\mathrm{s})/f(x|\mathrm{b})$ for x here



One possibility is to generate MC data and construct histograms for both signal and background.

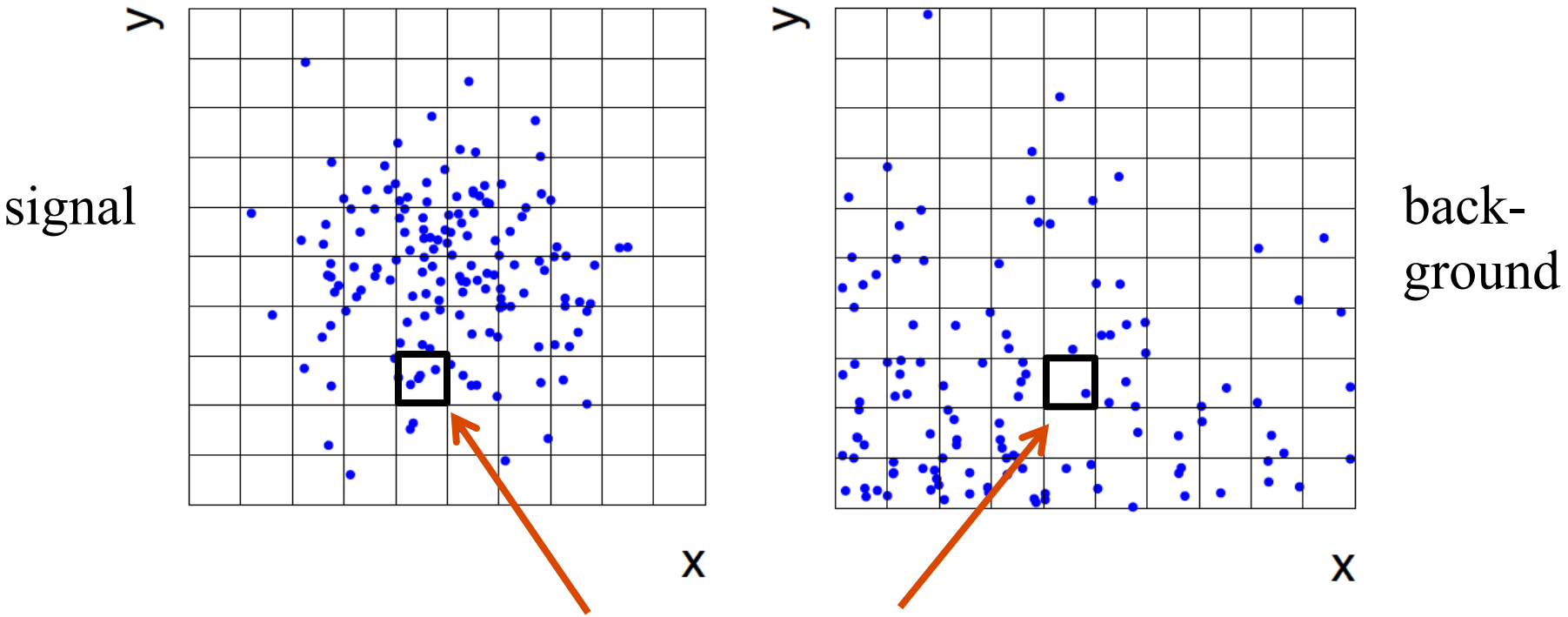
Use (normalized) histogram values to approximate LR:

$$t(x) \approx \frac{N(x|s)}{N(x|b)}$$

Can work well for single variable.

Approximate LR from 2D-histograms

Suppose problem has 2 variables. Try using 2-D histograms:



Approximate pdfs using $N(x,y|\mathrm{s})$, $N(x,y|\mathrm{b})$ in corresponding cells.

But if we want M bins for each variable, then in n-dimensions we have M^n cells; can't generate enough training data to populate.

→ Histogram method usually not usable for $n > 1$ dimension.

Strategies for multivariate analysis

Neyman-Pearson lemma gives optimal answer, but cannot be used directly, because we usually don't have $f(\boldsymbol{x}|\mathrm{s})$, $f(\boldsymbol{x}|\mathrm{b})$.

Histogram method with M bins for n variables requires that we estimate M^n parameters (the values of the pdfs in each cell), so this is rarely practical.

A compromise solution is to assume a certain functional form for the test statistic $t(\boldsymbol{x})$ with fewer parameters; determine them (using MC) to give best separation between signal and background.

Alternatively, try to estimate the probability densities $f(\boldsymbol{x}|\mathrm{s})$ and $f(\boldsymbol{x}|\mathrm{b})$ (with something better than histograms) and use the estimated pdfs to construct an approximate likelihood ratio.

Multivariate methods

Many new (and some old) methods esp. from Machine Learning:

- Fisher discriminant
- (Deep) neural networks
- Kernel density methods
- Support Vector Machines
- Decision trees
  - Boosting
  - Bagging

This is a large topic -- see e.g. lectures

http://www.pp.rhul.ac.uk/~cowan/stat/stat_2.pdf (from around p 38)

and references therein.

Testing significance / goodness-of-fit

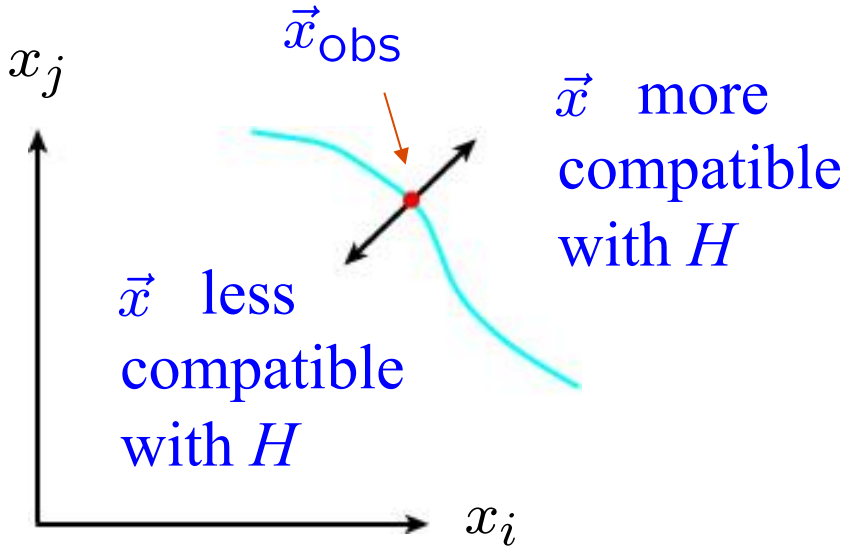
Suppose hypothesis H predicts pdf $f(\vec{x}|H)$ for a set of observations $\vec{x} = (x_1, \ldots, x_n)$.

We observe a single point in this space: $\vec{x}_{\text{obs}}$

What can we say about the validity of H in light of the data?

Decide what part of the data space represents less compatibility with H than does the point $\vec{x}_{\text{obs}}$.

This region therefore has greater compatibility with some alternative H'.



p-values

Express 'goodness-of-fit' by giving the *p*-value for *H*:

p = probability, under assumption of *H*, to observe data with equal or lesser compatibility with *H* relative to the data we got.

This is not the probability that *H* is true!

In frequentist statistics we don't talk about *P*(*H*) (unless *H* represents a repeatable observation). In Bayesian statistics we do; use Bayes' theorem to obtain

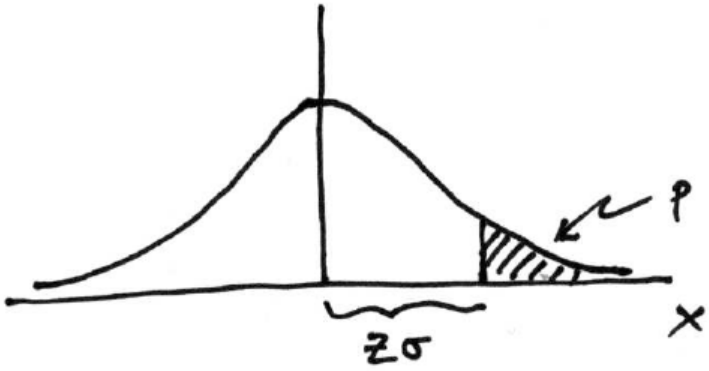
$$P(H|\vec{x}) = \frac{P(\vec{x}|H)\pi(H)}{\int P(\vec{x}|H)\pi(H)\,dH}$$

where $\pi(H)$ is the prior probability for *H*.

For now stick with the frequentist approach;
result is *p*-value, regrettably easy to misinterpret as *P*(*H*).

Significance from *p*-value

Often define significance Z as the number of standard deviations that a Gaussian variable would fluctuate in one direction to give the same p-value.



$$p = \int_Z^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 1 - \Phi(Z)$$ `1 - TMath::Freq`

$$Z = \Phi^{-1}(1-p)$$ `TMath::NormQuantile`

Test statistics and p-values

Consider a parameter μ proportional to rate of signal process.

Often define a function of the data (test statistic) q_μ that reflects level of agreement between the data and the hypothesized value μ.

Usually define q_μ so that higher values increasingly incompatibility with the data (more compatible with a relevant alternative).

We can define critical region of test of μ by $q_\mu \geq$ const., or equivalently define the p-value of μ as:

$$p_\mu = \int_{q_{\mu,\mathrm{obs}}}^{\infty} f(q_\mu|\mu)\, dq_\mu$$

observed value of q_μ

pdf of q_μ assuming μ

Equivalent formulation of test: reject μ if $p_\mu < \alpha$.

Confidence interval from inversion of a test

Carry out a test of size α for all values of μ.

The values that are not rejected constitute a *confidence interval* for μ at confidence level CL = $1 - \alpha$.

> The confidence interval will by construction contain the true value of μ with probability of at least $1 - \alpha$.

The interval will cover the true value of μ with probability $\geq 1 - \alpha$.

Equivalently, the parameter values in the confidence interval have p-values of at least α.

To find edge of interval (the "limit"), set $p_\mu = \alpha$ and solve for μ.

Coming up...

We now have most of the ingredients to carry out:

Parameter estimation

Statistical tests

Setting limits (confidence intervals)

In the second lecture we will apply these to some examples and extend the concepts e.g. to deal with systematic uncertainties.