Notes from Discussion Session 7 (16 Nov 20)

Prob Sheet 4

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1b) $r_i \sim \text{Uniform}[0,1]$ & indep.

Recall $E[r] = \frac{1}{2}$

$V[r] = \frac{1}{12}$

i) $x = r_1 + r_2 - 1$

$E[x] = E[r_1] + E[r_2] - 1 = 0$

$V[x] = V[r_1] + V[r_2] = \frac{1}{6}$

$\Rightarrow \sigma_x = \frac{1}{\sqrt{6}} = 0.408$

ii) $x = \sum_{i=1}^{4} r_i - 2$

$E[x] = \sum_{i=1}^{4} E[x_i] - 2 = 4 \times \frac{1}{2} - 2 = 0$

$V[x] = 4 \times \frac{1}{12} = \frac{1}{3}$, $\sigma_x = \frac{1}{\sqrt{3}} = 0.577$

iii) $x = \sum_{i=1}^{N} r_i - \frac{N}{2}$

$E[x] = N \times \frac{1}{2} - \frac{N}{2} = 0$

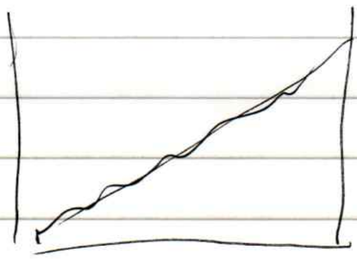
$V[x] = N \times \frac{1}{12} = \frac{N}{12} \rightarrow 1$ for $\underline{N=12}$

For large N, $x \sim$ Gauss because of CLT.

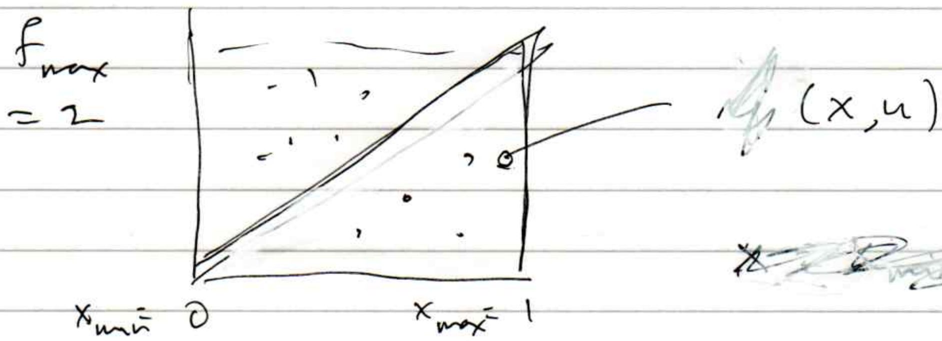
2a) $$f(x) = \begin{cases} \dfrac{2x}{x_{max}^2} & 0 < x < x_{max} \\ 0 & \text{otherwise} \end{cases}$$

$$F(x) = \int_0^x \frac{2x'}{x_{max}^2}\, dx' = \frac{x^2}{x_{max}^2} \overset{\text{set}}{=} r$$

$$\Rightarrow \quad x(r) = x_{max} \sqrt{r}$$



2b) Acc - rej method



$$x = x_{min} + (x_{max} - x_{min})\, \underline{r_1}$$

$$u = \underline{r_2} \cdot f_{max}$$

If $u < f(x)$, accept $\underline{x}$.

3) $\vec{x} \sim \text{Gauss}(\vec{\mu}_k, \underbrace{V}_{\text{cov. matrix}}) \qquad k = 0, 1$

i.e. $$f(\vec{x} \mid \vec{\mu}_k) = \frac{1}{(2\pi)^{n/2} |V|^{1/2}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu}_k)^T V^{-1} (\vec{x}-\vec{\mu}_k)\right]$$

Let test statistic $t(\vec{x}) = \ln \dfrac{f(\vec{x} \mid \vec{\mu}_1)}{f(\vec{x} \mid \vec{\mu}_0)}$

a) $$\ln \frac{f(\vec{x} \mid \vec{\mu}_1)}{f(\vec{x} \mid \vec{\mu}_0)} = -\frac{1}{2}\left[(\vec{x}-\vec{\mu}_1)^T V^{-1}(\vec{x}-\vec{\mu}_1) - (\vec{x}-\vec{\mu}_0)^T V^{-1}(\vec{x}-\vec{\mu}_0)\right]$$

$$= -\frac{1}{2}\Big[\cancel{\vec{x}^T V^{-1} \vec{x}} - \vec{\mu}_1^T V^{-1} \vec{x} - \vec{x}^T V^{-1} \vec{\mu}_1 + \vec{\mu}_1^T V^{-1} \vec{\mu}_1$$

$$- \cancel{\vec{x}^T V^{-1} \vec{x}} + \vec{\mu}_0^T V^{-1} \vec{x} + \underbrace{\vec{x}^T V^{-1} \vec{\mu}_0}_{= \vec{\mu}_0^T V^{-1} \vec{x}} - \vec{\mu}_0^T V^{-1} \vec{\mu}_0\Big]$$

since scalar $(\,) = (\,)^T$ and $V^{-1} = (V^{-1})^T$

$$= \underbrace{-\frac{1}{2}\left[\vec{\mu}_1^T V^{-1} \vec{\mu}_1 - \vec{\mu}_0 V^{-1} \vec{\mu}_0\right]}_{w_0} + \underbrace{(\vec{\mu}_1 - \vec{\mu}_0)^T V^{-1}}_{\vec{w}^T} \vec{x}$$

b)

$$= w_0 + \vec{w}^T \vec{x}$$

$$\vec{w} = \left[(\vec{\mu}_1 - \vec{\mu}_0)^T V^{-1}\right]^T = \underbrace{(V^{-1})^T}_{= V^{-1}} (\vec{\mu}_1 - \vec{\mu}_0)$$

If $W = V + V \Rightarrow W^{-1} = \frac{1}{2} V^{-1}$

$$\Rightarrow \vec{w} = 2W^{-1}(\vec{\mu}_1 - \vec{\mu}_0)$$

c) Find $P(H_0 | \vec{x})$

$$= \frac{f(\vec{x} | H_0)\pi_0}{f(\vec{x} | H_0)\pi_0 + f(\vec{x} | H_1)\pi_1}$$

$$= \frac{\pi_0}{\pi_0 + \pi_1 e^{t}} \qquad t = \frac{f(\vec{x} | H_1)}{f(\vec{x} | H_0)}$$

$$= \frac{1}{1 + \frac{\pi_1}{\pi_0} e^{t}}$$

Note on Extended Maximum Likelihood

$$n \sim \text{Poisson}(\nu)$$

$$x \sim f(x;\theta) \rightarrow x_1, \ldots, x_n$$

$$P(n, \vec{x}) = P(\vec{x} \mid n)\, P(n) = L(\nu, \theta)$$

$$= \underbrace{\frac{\nu^n}{n!} e^{-\nu}}_{P(n)} \underbrace{\prod_{i=1}^{n} f(x_i;\theta)}_{P(\vec{x} \mid n)}$$

$$\ln L(\nu,\theta) = n \ln \nu - \nu + \sum_{i=1}^{n} \ln f(x_i;\theta) + C$$

$$\frac{\partial \ln L}{\partial \nu} = \frac{n}{\nu} - 1 \overset{\text{set}}{=} 0 \Rightarrow \underline{\hat{\nu} = n}$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{\partial}{\partial \theta}\Bigg(\underbrace{\sum_{i=1}^{n} \ln f(x_i;\theta)}_{\text{same as usual ML}} \Bigg) \overset{\text{set}}{=} 0 \Rightarrow \hat{\theta}_{EML} = \hat{\theta}_{ML}$$

So $\hat{\nu} = n$ & $\hat{\theta}$ same as if n fixed. But now $V[\hat{\theta}]$ is increased since allowing n to vary introduces an additional element of fluctuation.