Simplified "Errors on Errors" Model

The model in Lectures 11-3, 11-4

Details in: G. Cowan, *Statistical Models with Uncertain Error Parameters*, Eur. Phys. J. C (2019) 79:133, arXiv:1809.05778

makes a distinction between the $\sigma_{y,i}$ (~statistical errors), which are known, and the $\sigma_{u,i}$ ~systematic errors), which are treated as adjustable parameters.

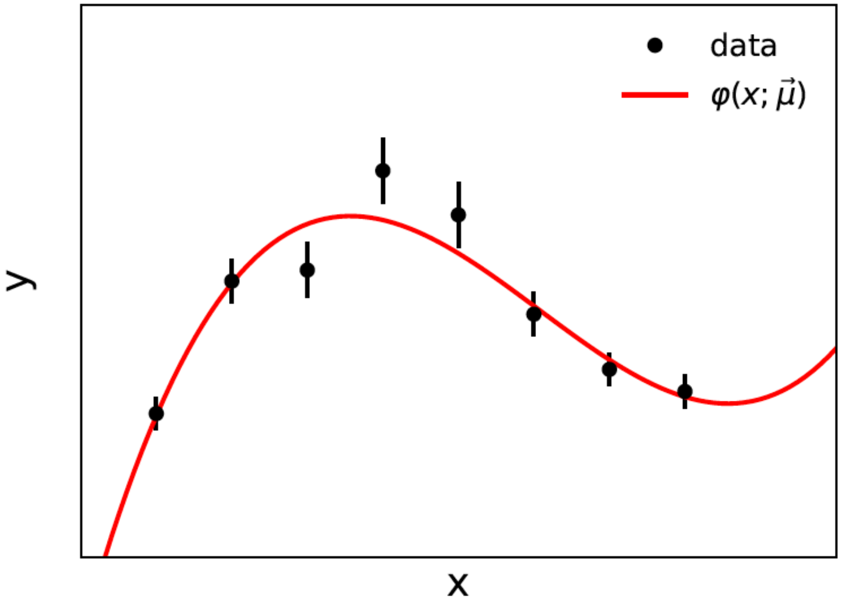
Here we show a simplified model that does not distinguish between statistical and systematic errors.

Curve fitting, averages

Suppose independent
$y_i \sim$ Gauss, $i = 1,\ldots,N$, with

$$E[y_i] = \varphi(x_i; \boldsymbol{\mu})$$

$$V[y_i] = \sigma_i^2$$



$\boldsymbol{\mu}$ are the parameters in the fit function $\varphi(x;\boldsymbol{\mu})$.

If we take the σ_i as known, we have the usual log-likelihood

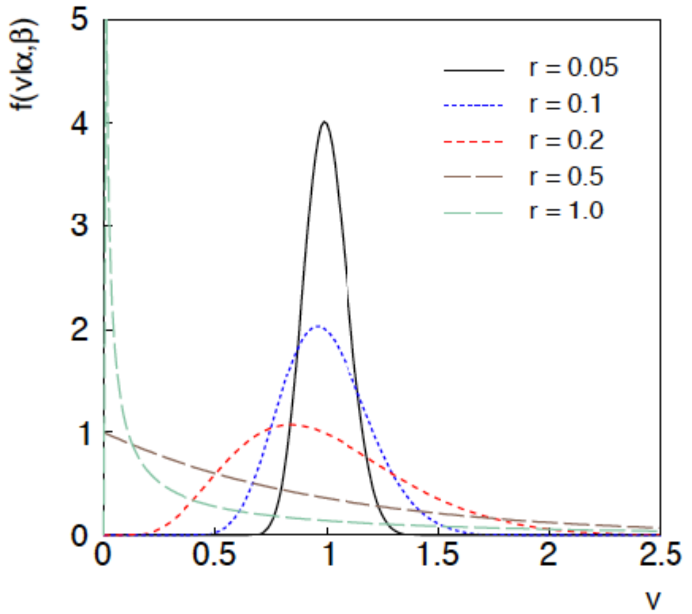
$$\ln L(\boldsymbol{\mu}) = -\frac{1}{2} \sum_{i=1}^{N} \frac{(y_i - \varphi(x_i; \boldsymbol{\mu}))^2}{\sigma_i^2}$$

which leads to the Least Squares estimators for $\boldsymbol{\mu}$.

Model with uncertain σ_i^2

If the σ_i^2 are uncertain, we can take them as adjustable parameters.

The estimated variances $v_i = s_i^2$ are modeled as gamma distributed.



The likelihood becomes

$$L(\boldsymbol{\mu}, \boldsymbol{\sigma^2}) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-(y_i - \varphi(x_i;\boldsymbol{\mu}))^2/2\sigma_i^2} \, \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} v_i^{\alpha_i - 1} e^{-\beta_i v_i}$$

Want $\quad E[v_i] = \sigma_i^2 \qquad \dfrac{\sigma_{s_i}}{E[s_i]} \approx r_i \qquad (s_i = \sqrt{v_i})$

$\rightarrow \quad \alpha_i = \dfrac{1}{4r_i^2} \qquad \beta_i = \dfrac{\alpha_i}{\sigma_i^2}$

Profile log-likelihood

One can profile over the σ_i^2 in close form.

The log-profile-likelihood is

$$\ln L'(\boldsymbol{\mu}) = \ln L(\boldsymbol{\mu}, \widehat{\widehat{\boldsymbol{\sigma}^2}}) = -\frac{1}{2} \sum_{i=1}^{N} \left(1 + \frac{1}{2r_i^2} \right) \ln \left[1 + 2r_i^2 \frac{(y_i - \varphi(x_i; \boldsymbol{\mu}))^2}{v_i} \right]$$

Quadratic terms replace by sum of logs.

Equivalent to replacing Gauss pdf for y_i by Student's t, $\nu_{\rm dof} = 1/2r_i^2$

Confidence interval for $\boldsymbol{\mu}$ becomes sensitive to goodness-of-fit (increases if data internally inconsistent).

Fitted curve less sensitive to outliers.

Simple program for Student's t average: stave.py