

Variable transformation w/ δ function
 cf. D. Gillespie, Am. J. Phys. 51 (1983) 520,
 Harrison Prosper INFN School of Statistics
 lectures on probability (2024)

Theorem: $\vec{x} \sim f(\vec{x})$ $\vec{x} = (x_1, \dots, x_n)$
 $\vec{y} = \vec{a}(\vec{x})$ $\vec{y} = (y_1, \dots, y_m)$
 proof

pdf $\rightarrow$
$$g(\vec{y}) = \int d^n \vec{x} f(\vec{x}) \delta^{(m)}(\vec{y} - \vec{a}(\vec{x}))$$

Example: $f(x, y) = 1$ $0 \leq x \leq 1, 0 \leq y \leq 1$

$$z = xy$$

$$g(z) = \iint dx dy f(x, y) \delta(z - xy)$$

$\hookrightarrow h(y) \equiv z - xy$

Use $\delta(h(y)) = \frac{\delta(y - y_0)}{|h'(y_0)|}$, $y_0 = \text{zero of } h$

$$h(y_0) = z - xy_0 = 0 \Rightarrow y_0 = \frac{z}{x}, h'(y_0) = -x$$

$$g(z) = \int dx \int dy f(x, y) \frac{\delta(y - \frac{z}{x})}{|-x|}$$

$$= \int f(x, \frac{z}{x}) \frac{dx}{x}$$

← same as before

Discussion Session - week 2

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Example 1

Consider the joint pdf

$$f(x, y) = 1, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1$$

a) Find pdf of $z = xy$

In lectures we found

$$g(z) = \int f\left(x, \frac{z}{x}\right) \frac{dx}{x} \quad (\text{Mellin convolution})$$

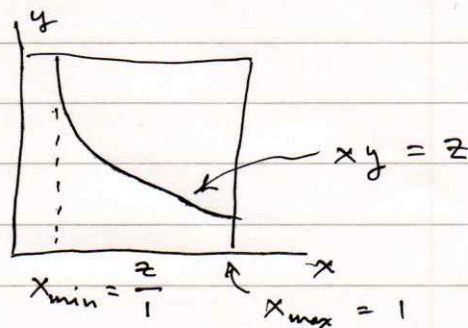
$$= \int_{x_{\min}}^{x_{\max}} 1 \cdot \frac{dx}{x}$$

 $f(x, y)$ is nonzero for $0 \leq x \leq 1, 0 \leq y \leq 1$

$$\Rightarrow 0 \leq \frac{z}{x} \leq 1 \quad \Rightarrow \quad 0 \leq z \leq x$$

$$\Rightarrow x_{\min} = z$$

$$x_{\max} = 1$$



$$\Rightarrow g(z) = \int_z^1 \frac{dx}{x} = \ln x \Big|_z^1$$

$$= -\ln z, \quad 0 < z \leq 1$$

b) Alternative method. - let

$$z = xy$$

$\Rightarrow$

$$x = u$$

$$u = x$$

$$y = \frac{z}{u}$$

Jacobian is

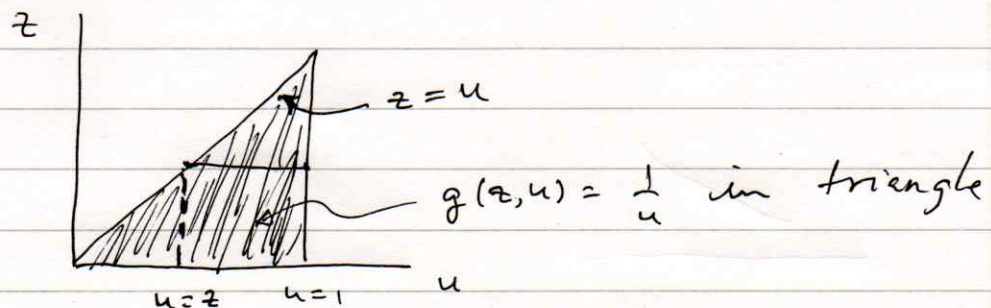
$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{u} & -\frac{z}{u^2} \end{vmatrix} = -\frac{1}{u}$$

$$g(z, u) = |J| f(x(z, u), y(z, u))$$

$$= \frac{1}{u}, \quad 0 \leq u \leq 1, \quad 0 \leq z \leq u$$

Because: $0 \leq x \leq 1 \Rightarrow 0 \leq u \leq 1$

$$0 \leq y \leq 1 \Rightarrow 0 \leq \frac{z}{u} \leq 1 \Rightarrow 0 \leq z \leq u$$



$$g_z(z) = \int g(z, u) du = \int_z^1 \frac{du}{u} = -\ln z, \quad 0 < z \leq 1$$

$\uparrow$ $u \geq z$ (see above)

Example 2 - error propagation

Consider r.v.s x_i , $i=1, 2$

with $\mu_i = 10$, $\sigma_i = 1$, $\text{cov}[x_i, x_j] = 0$

and let $y = \frac{x_1^2}{x_2}$. Find variance $V[y]$.

$$V[y] \approx \left(\frac{\partial y}{\partial x_1} \right)^2 \bigg|_{\vec{x}=\vec{\mu}} \sigma_1^2 + \left(\frac{\partial y}{\partial x_2} \right)^2 \bigg|_{\vec{x}=\vec{\mu}} \sigma_2^2$$

$$= \left(\frac{2x_1}{x_2} \right)^2 \bigg|_{\vec{x}=\vec{\mu}} \sigma_1^2 + \left(-\frac{x_1^2}{x_2^2} \right)^2 \bigg|_{\vec{x}=\vec{\mu}} \sigma_2^2$$

$$= \frac{4\mu_1^2}{\mu_2^2} \sigma_1^2 + \frac{\mu_1^4}{\mu_2^4} \sigma_2^2$$

$$= 4 \times 1 + 1 \times 1 = 5 \Rightarrow \sigma_y = \sqrt{5} \approx 2.2$$

Now suppose $\mu_1 = 10$, $\mu_2 = 1$ ($\sigma_1 = \sigma_2 = 1$)

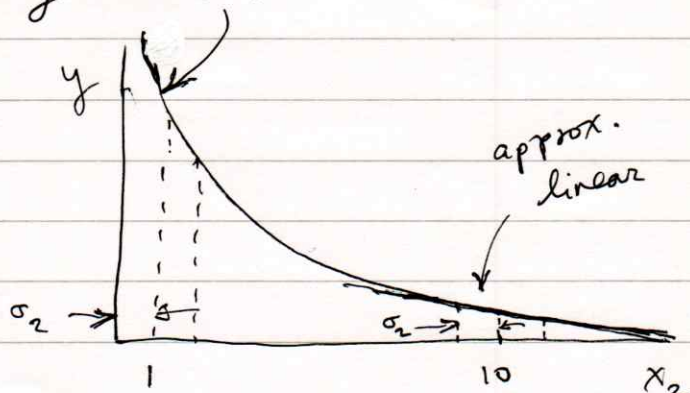
Then $y = \frac{x_1^2}{x_2}$ is significantly non-linear in

a region of $\sim \pm \sigma_2$

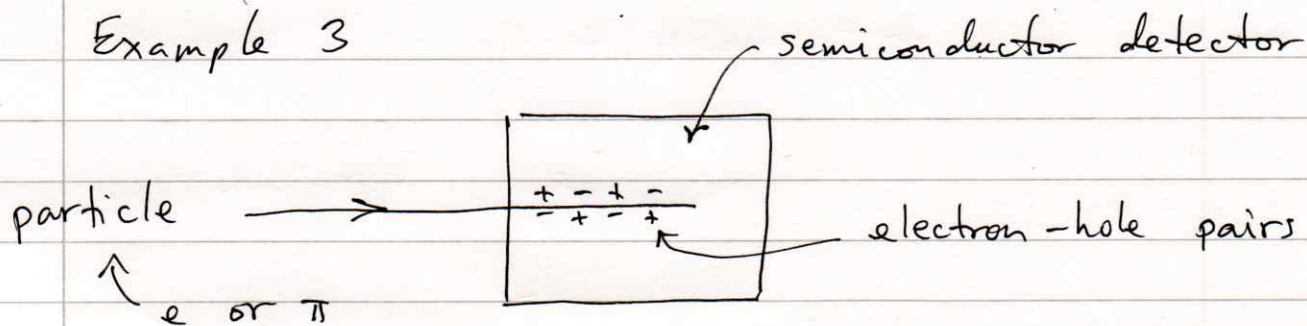
& therefore linear err. prop.

is poor approx.

$$(\sigma_y \rightarrow \sqrt{10400} = 102.0)$$



Example 3



of e^- /hole pairs n $\sim \text{Poisson}(\nu_e)$ for e
 $\sim \text{Poisson}(\nu_\pi)$ for π

Incident particles are mixture of e, π

w/ relative fractions π_e, π_π (prior probs.)
 $\uparrow = 1 - \pi_e$

From law of total probability

$$P(n) = P(n|\nu_\pi) \pi_\pi + P(n|\nu_e) \pi_e$$

$$= \frac{\nu_\pi^n}{n!} e^{-\nu_\pi} \pi_\pi + \frac{\nu_e^n}{n!} e^{-\nu_e} \pi_e$$

Expectation value of n is

$$E[n] = \sum_{n=0}^{\infty} n P(n)$$

$$= \pi_\pi \underbrace{\sum_{n=0}^{\infty} n P(n|\nu_\pi)}_{= E[n|\nu_\pi] = \nu_\pi} + \pi_e \underbrace{\sum_{n=0}^{\infty} n P(n|\nu_e)}_{= E[n|\nu_e] = \nu_e}$$

$$= \pi_\pi \nu_\pi + \pi_e \nu_e$$

Find variance $V[n] = E[n^2] - (E[n])^2$

First find

$$E[n^2] = \sum_{n=0}^{\infty} n^2 \left(P(n|\lambda_{\pi}) \pi_{\pi} + P(n|\lambda_e) \pi_e \right)$$

$$= \pi_{\pi} E[n^2 | \lambda_{\pi}] + \pi_e E[n^2 | \lambda_e]$$

Use fact that

$$E[n^2] = V[n] + (E[n])^2$$

and for Poisson var. $V[n] = E[n]$

$$\Rightarrow E[n^2 | \lambda_i] = \lambda_i + \lambda_i^2, \quad i = \pi, e$$

Assembling the ingredients,

$$V[n] = \pi_{\pi} (\lambda_{\pi} + \lambda_{\pi}^2) + \pi_e (\lambda_e + \lambda_e^2)$$

$$- (\pi_{\pi} \lambda_{\pi} + \pi_e \lambda_e)^2$$

Example 4 - proof that covariance matrix

$V_{ij} = \text{cov}[x_i, x_j]$ is positive semi-definite

i.e. $\vec{z}^T V \vec{z} \geq 0$ for any $\vec{z} \in \mathbb{R}^n$

Can transform r.v.s to have mean of zero

i.e. let $x_i \rightarrow x_i - \mu_i$ so that

$$V = E[\vec{X} \vec{X}^T], \quad \vec{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

↑ "outer product"

$$\text{Let } \vec{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n \quad (\text{const.})$$

$$\vec{z}^T V \vec{z} = \vec{z}^T E[\vec{X} \vec{X}^T] \vec{z}$$

$$= E[\vec{z}^T \vec{X} \vec{X}^T \vec{z}] \quad \text{since } E[\] \text{ linear}$$

$$= E[(\vec{X}^T \vec{z})^T (\vec{X}^T \vec{z})] \quad \text{since } A^T B = ((A^T B)^T)^T$$

$$= E[(\vec{z}^T \vec{X})^2] \geq 0 \quad = (B^T A)^T$$

↑ real scalar

Q.E.D.

$$\text{For e.g. } z_i = \delta_{ij} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \text{ at position } j$$

$$\Rightarrow E[x_j^2] = V[x_j] \geq 0.$$