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SDA PH4515 Week 8 Discussion Notes

From 2016 Exam Q5, averaging measuments

$$y_i \sim \text{Gauss}(\lambda, \sigma_i) \qquad i=1,2$$

↳ indep. param to be measured.

a) Likelihood function

$$L(\lambda) = \prod_{i=1}^{2} \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-(y_i-\lambda)^2/2\sigma_i^2}$$

$$\Rightarrow \ln L(\lambda) = -\frac{1}{2}\sum_{i=1}^{2} \frac{(y_i-\lambda)^2}{\sigma_i^2} + C$$

$$\Rightarrow \text{minimize} \quad \chi^2(\lambda) = \sum_{i=1}^{2} \frac{(y_i-\lambda)^2}{\sigma_i^2}$$

b) $$\frac{\partial \chi^2}{\partial \lambda} = -\frac{2(y_1-\lambda)}{\sigma_1^2} - \frac{2(y_2-\lambda)}{\sigma_2^2} \overset{\text{set}}{=} 0$$

$$\Rightarrow \hat{\lambda} = \frac{y_1/\sigma_1^2 + y_2/\sigma_2^2}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}$$

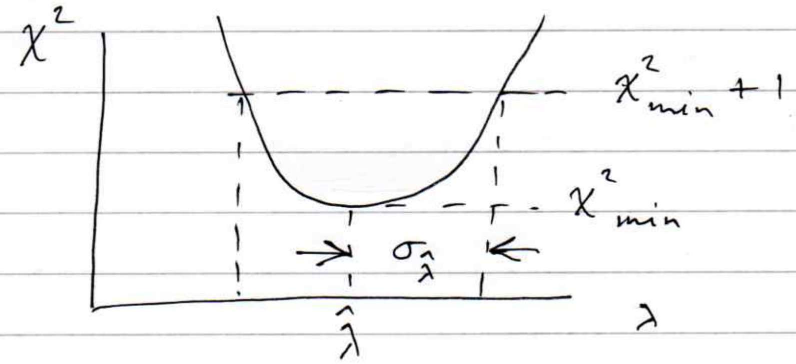
$$= \underbrace{\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}}_{= w} y_1 + \underbrace{\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}}_{= 1-w} y_2$$

c) $$E[\hat{\lambda}] = \frac{\frac{E[y_1]}{\sigma_1^2} + \frac{E[y_2]}{\sigma_2^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}\,, \quad \text{use } E[y_i] = \lambda \quad i = 1, 2$$

$$= \lambda \Rightarrow \hat{\lambda} \text{ is unbiased.}$$

$$V[\hat{\lambda}] = \frac{1}{\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^2} \left(\frac{\overset{\sigma_1^2}{V[y_1]}}{\sigma_1^4} + \frac{\overset{\sigma_2^2}{V[y_2]}}{\sigma_2^4} \right) = \frac{1}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}$$

d)



$$p = \int_{\chi^2_{min}}^{\infty} f_{\chi^2}(t; \underset{\text{2 measurements} - \text{1 fitted param.}}{n_d = 1})\, dt$$

e) Suppose $\hat{\lambda} = w y_1 + (1-w) y_2$ for some w

= most general linear unbiased λ

$$E[\hat{\lambda}] = w \underset{\lambda}{E[y_1]} + (1-w) \underset{\lambda}{E[y_2]} = \lambda \Rightarrow \text{unbiased} \checkmark$$

$$V[\hat{\lambda}] = w^2 \sigma_1^2 + (1-w)^2 \sigma_2^2 \rightsquigarrow \text{adjust } w \text{ to minimize}$$

$$\frac{\partial V[\hat{\lambda}]}{\partial w} = 2 w \sigma_1^2 - 2(1-w)\sigma_2^2 \overset{\text{set}}{=} 0$$

$$\Rightarrow w = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \Rightarrow \text{same as LS!}$$

$\hat{\lambda}$ = "BLUE" (Best Linear Unbiased Estimator)