Notes from Statistics Online Session #2 (12.10.20)

From 2019 Exam #1:

Suppose x, y indep. & follow Gauss pdf w/ mean 0, $\sigma = 1$

$$f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad -\infty < x < \infty$$

$$f_y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \qquad -\infty < y < \infty$$

$$\rightarrow f(x,y) = f_x(x) f_y(y)$$

Let $z = \frac{x}{y}$, $u = x$

$\Rightarrow$ $x = u$, $y = \frac{u}{z}$ } invert transform.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{z} & -\frac{u}{z^2} \end{vmatrix} = -\frac{u}{z^2}$$

$$\Rightarrow g(u,z) = |J| \, f(x(u,z), y(u,z))$$

$$= \frac{|u|}{z^2} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \frac{1}{\sqrt{2\pi}} e^{-u^2/2z^2}$$

$$= \frac{|u|}{2\pi z^2} \exp\left[-\frac{u^2}{2}\left(\frac{z^2+1}{z^2}\right)\right] \qquad \begin{matrix} -\infty < u < \infty \\ -\infty < z < \infty \end{matrix}$$

↳ does not factorize $\Rightarrow$ u, z not independent

Find marginal pdfs

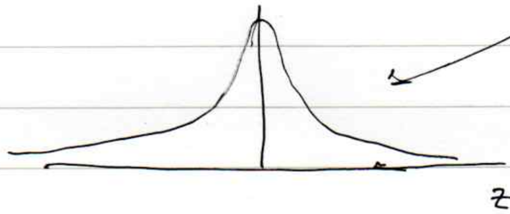
$$g_z(z) = \int_{-\infty}^{\infty} g(u,z)\,du = \int_{-\infty}^{\infty} \frac{|u|}{2\pi z^2} \exp\left[-\frac{u^2}{2}\left(\frac{z^2+1}{z^2}\right)\right] du$$

integrand even

$$= 2\int_0^{\infty} \sim du$$

Use $\int_0^{\infty} x e^{-ax^2} dx = \frac{1}{2a}$, $a = \frac{z^2+1}{2z^2}$

$$\Rightarrow g_z(z) = \frac{2}{2\pi z^2} \times \frac{2z^2}{2(1+z^2)} = \underline{\frac{1}{\pi}\frac{1}{1+z^2}}$$



Cauchy pdf

Try to find variance.

Suppose $E[z] = 0$?

$$V[z] = E[z^2] - (\cancel{E[z]})^2$$

$$= \int_{-\infty}^{\infty} z^2 \frac{1}{\pi}\frac{1}{1+z^2}\,dz$$

integrand $\rightarrow \frac{1}{\pi}$ for $z \rightarrow \pm\infty$

$\Rightarrow$ integral diverges

Variance does not exist

Find conditional pdf

$$g(u|z) = \frac{g(u,z)}{g_z(z)}$$

$$= \frac{u}{2\pi z^2} \exp\left[-\frac{u^2(1+z^2)}{2z^2}\right] \times \pi(1+z^2)$$

$$= \frac{|u|(1+z^2)}{2z^2} \exp\left[-\frac{u^2(1+z^2)}{2z^2}\right] \qquad -\infty < u < \infty$$

i.e. imposing a value of z affects pdf of u $\Rightarrow$ u, z not independent

$$g(z|u) = \frac{g(u,z)}{g_u(u)}.$$

$u = x$ so $g_u(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$

$$\Rightarrow g(z|u) = \frac{\frac{|u|}{2\pi z^2} \exp\left[-\frac{u^2(1+z^2)}{2z^2}\right]}{\frac{1}{\sqrt{2\pi}} e^{-u^2/2}}$$

$$= \frac{|u|}{\sqrt{2\pi}\, z^2} e^{-u^2/2z^2}$$

Same result from Bayes' theorem:

$$g(z|u) = \frac{g(u/z)\, g_z(z)}{g_u(u)}$$

Suppose $E[x] \to \mu_x$, $E[y] \to \mu_y$, e.g.

$$f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-(x-\mu_x)^2/2\sigma_x^2}$$

$$f_y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-(y-\mu_y)^2/2\sigma_y^2}$$

indep.

As before, let $z = \frac{x}{y}$,

$$V[z] \approx \left(\frac{\partial z}{\partial x}\right)^2\Big|_{\vec{\mu}} \sigma_x^2 + \left(\frac{\partial z}{\partial y}\right)^2\Big|_{\vec{\mu}} \sigma_y^2$$ (err. prop.)

$$\frac{\partial z}{\partial x} = \frac{1}{y} \to \frac{1}{\mu_y}$$

$$\frac{\partial z}{\partial y} = -\frac{x}{y^2} \to \frac{\mu_x}{\mu_y^2}$$

$$\Rightarrow V[z] = \frac{\sigma_x^2}{\mu_y^2} + \frac{\mu_x^2}{\mu_y^4}\sigma_y^2$$

For $\mu_y \to 0$, $V[z] \to \infty$ as found before

Error prop. approx. ok if function ~linear

$\Rightarrow$ need $\sigma_y \ll \mu_y$

($z = \frac{x}{y}$ anyway exactly linear in x.)