2019 Exam #1

12 Oct 20
Disc. Session

$$\begin{cases} x \sim f_x(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \\ y \sim f_y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \end{cases}$$

$f(x; \mu, \sigma)$

$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$

$\rightarrow \mu = 0 \;, \quad \sigma = 1$

joint pdf $\quad f(x,y) = f_x(x)\, f_y(y) \Rightarrow$ x, y independent.

Let $\underline{z} = \frac{x}{y} \;, \quad \underline{u} = x$

goal: find $g(z, u)$

$\Rightarrow \quad \underline{x} = u \quad , \quad \underline{y} = \dfrac{u}{z}$

$$g(u, z) = |J| \; f(x(u, z), y(u, z))$$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial z} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\[2ex] \dfrac{1}{z} & -\dfrac{u}{z^2} \end{vmatrix} = -\dfrac{u}{z^2}$$

$$= \frac{|u|}{z^2} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \times \frac{1}{\sqrt{2\pi}} e^{-u^2/2z^2}$$

$$g(u,z) = \frac{|u|}{2\pi z^2} \exp\left[-\frac{u^2}{2}\left(\frac{z^2+1}{z^2}\right)\right]$$

does not factorize

$\rightarrow$ u, z <u>not</u> independent.

$u = x$

$z = \frac{x}{y}$

$\Rightarrow -\infty < u < \infty$

$-\infty < z < \infty$

$$g_z(z) = \int g(u,z)\, du$$

$$= \int_{-\infty}^{\infty} \frac{|u|}{2\pi z^2} \exp\left[-\frac{u^2}{2}\left(\frac{z^2+1}{z^2}\right)\right] du$$

integrand even in u

$$= 2\int_0^{\infty} \text{———}\, du$$

$$g_z(z) = \frac{1}{\pi}\frac{1}{1+z^2} \longleftarrow \text{Cauchy}$$

"long" tails

$$V[z] = E[z^2] - \big(\overset{0}{\cancel{E[z]}}\big)^2$$

$$E[z] = \lim_{\substack{a\to-\infty,\\ b\to\infty}} \int_a^b z \,\frac{1}{\pi}\cancel{\frac{1}{1+z^2}}\, dz \longrightarrow 0 \;?$$

$$= \int_a^0 + \int_0^b$$

$V[z]$ not defined.

$$V[z] = \int_{-\infty}^{\infty} \underbrace{z^2 \,\frac{1}{\pi}\frac{1}{1+z^2}}_{\to \frac{1}{\pi} \text{ fn } z \to \pm\infty} dz \longrightarrow \infty$$

$$g(u|z) = \frac{g(u,z)}{g_z(z)}$$

$$= \frac{|u|}{2\pi z^2} \exp\left[-\frac{u^2(1+z^2)}{2z^2}\right] \cdot \pi(1+z^2)$$

$$-\infty < u < \infty.$$

$\Rightarrow$ imposing z affects pdf of u

$\Rightarrow$ u, z <u>not</u> indep.

$$g(z|u) = \frac{g(u,z)}{g_u(u)}$$

$$g_u(u) = \int g(u,z)\, dz$$

$$= \frac{1}{\sqrt{2\pi}} e^{-u^2/2}$$

$$= \frac{|u|}{\sqrt{2\pi}\, z^2} e^{-u^2/2z^2}$$ ✓

$u = x$

$z = \frac{x}{y}$

Def. $g(u|z)$

~~tan~~ g

Or, Bayes' thm: $g(z|u) = \frac{g(u|z)\, g_z(z)}{g_u(u)}$

$$f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-(x-\mu_x)^2/2\sigma_x^2}$$

$$f_y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-(y-\mu_y)^2/2\sigma_y^2}$$

indep.

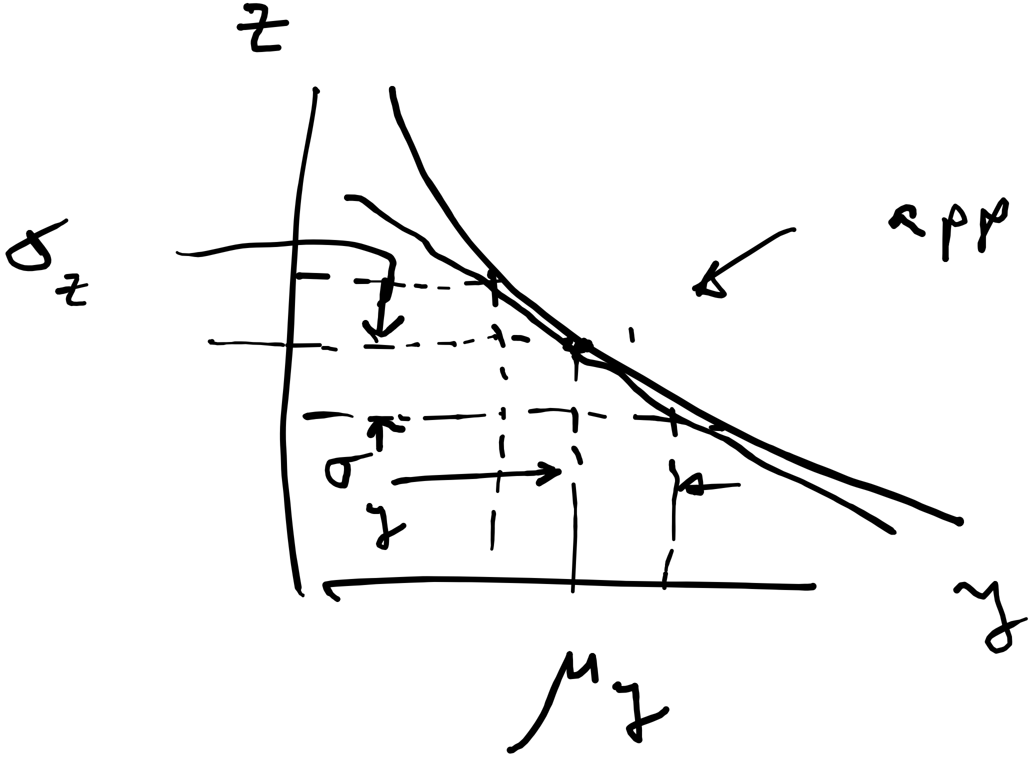
$$z = \frac{x}{y}$$

$$V[z] = \left(\frac{\partial z}{\partial x}\right)^2\Big|_{\vec{\mu}} \sigma_x^2 + \left(\frac{\partial z}{\partial y}\right)^2\Big|_{\vec{\mu}} \sigma_y^2$$

$$\underbrace{+ 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\cancel{\operatorname{cov}[x,y]}}_{0} = \sum_{i} \frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}\operatorname{cov}[x_i\, x_j]$$

$$\rightarrow \frac{\sigma_x^2}{\mu_y^2} + \frac{\mu_x^2}{\mu_y^4}\sigma_y^2 \rightarrow \infty \text{ when } \mu_y \rightarrow 0$$

Error prop. based on linear approx



approx ok

$z = \frac{x}{y}$

if $z(y) \approx$ linear

$\mu_y \pm \sigma_y$