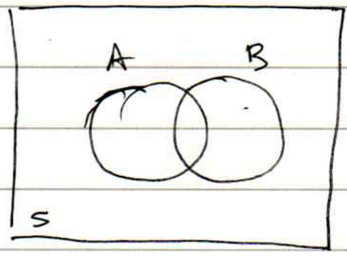
Notes for discussion session Week 4 (26 Oct 20)

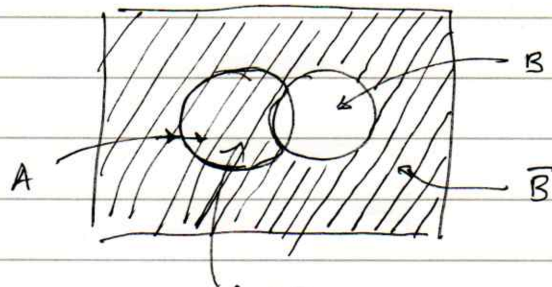
G. COWAN
1

1) Show $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Recall

$A - B \equiv A \cap \overline{B}$



$$A \cup B = \underbrace{(A - A \cap B) \cup B}_{\text{disjoint}}$$

$$\Rightarrow P(A \cup B) = P(A - A \cap B) + P(B) \quad (1)$$

also, $A = (A - A \cap B) \cup A \cap B$ (disjoint)

$$\Rightarrow P(A) = P(A - A \cap B) + P(A \cap B) \quad (2)$$

Use (1) + (2) to eliminate $P(A - A \cap B)$

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

2) Use Baye's theorem

$$P(A|B) = \frac{P(B|A)\,P(A)}{\sum_i P(B|A_i)\,P(A_i)}$$

a) $$P(\gamma|1) = \frac{P(1|\gamma)\,P(\gamma)}{P(1|\gamma)\,P(\gamma) + P(1|e)\,P(e)}$$

$$= \frac{0.001 \times 0.9999}{0.001 \times 0.9999 + 0.01 \times 10^{-4}}$$

$$= 0.999000899$$

b) $$P(e|2) = \frac{P(2|e)\,P(e)}{P(2|e)\,P(e) + P(2|\gamma)\,P(\gamma)}$$

$$= \frac{0.989 \times 0.0001}{0.989 * 0.0001 + 10^{-5}(1 - 0.0001)}$$

$$\simeq 0.90818$$

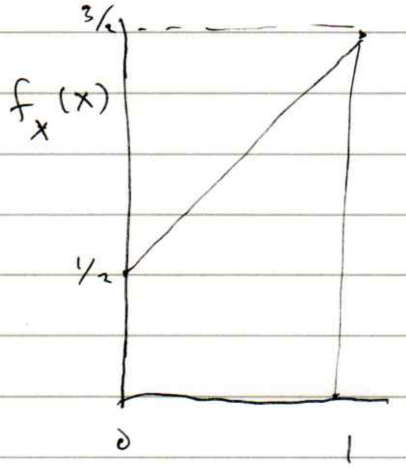
3) $$f(x,y) = \begin{cases} x + y & 0 \leq x \leq 1,\ 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

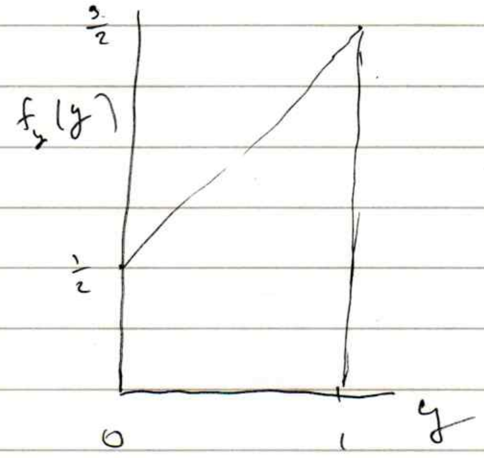
a) Find marginal pdfs

$$f_x(x) = \int_0^1 f(x,y)\,dy = \int_0^1 (x+y)\,dy$$

$$= x + \frac{1}{2} \qquad 0 \leq x \leq 1$$

By symmetry $f_y(y) = y + \frac{1}{2} \qquad 0 \leq y \leq 1$





(same)

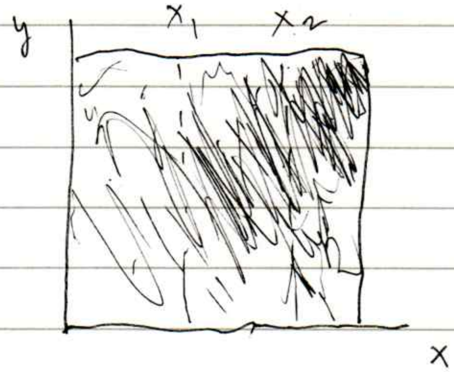
x & y <u>not</u> independent because

$$f(x,y) \neq f_x(x)\,f_y(y)$$

Also consider e.g.

$$f(y|x_1) \neq f(y|x_2)$$

etc



b) From definition of conditional prob

$$f(x|y) = \frac{f(x,y)}{f_y(y)} = \frac{x+y}{\frac{1}{2}+y} \qquad 0 \le x \le 1$$

+ by symmetry

$$f(y|x) = \frac{x+y}{\frac{1}{2}+x} \qquad 0 \le y \le 1$$

According to Bayes' thm these are related by

$$f(x|y) = \frac{f(y|x)\, f_x(x)}{f_y(y)}$$

Substituting the ingredients

$$\frac{x+y}{\frac{1}{2}+y} \stackrel{?}{=} \frac{(x+y)\cancel{(\frac{1}{2}+x)}}{\cancel{(\frac{1}{2}+x)}(\frac{1}{2}+y)} \qquad \checkmark$$

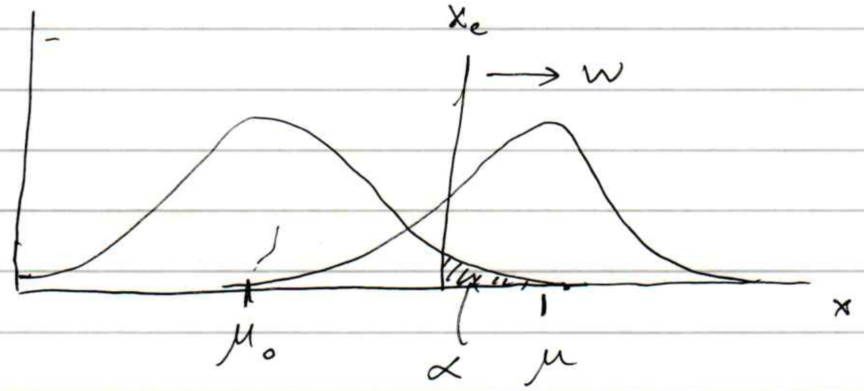
which holds, as required.

From Week 4 extra slides

$x \sim \text{Gauss}(\mu, \sigma)$ ← μ: want to test, σ: known

$H_0 : \mu = \mu_0$ vs. $H_1 : \mu > \mu_0$

Take $W = \{ x : x \geq x_c \}$



$$\alpha = P(x \geq x_c \mid \mu_0)$$

$$= \int_{x_c}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu_0)^2/2\sigma^2} dx$$

let $y = \dfrac{x - \mu_0}{\sigma}$

$$= \int_{\frac{x_c - \mu_0}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} e^{-y^2/2} \cdot \cancel{\sigma}\, dy$$

$$= 1 - \Phi\left(\frac{x_c - \mu_0}{\sigma}\right)$$

↑ standard Gauss. cumul. dist.

$$\Rightarrow x_c = \mu_0 + \sigma\,\Phi^{-1}(1 - \alpha)$$

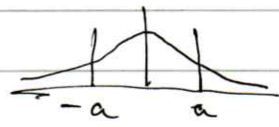
↑ Std. Gauss. Quantile

$$\text{Power} = M = P(x \geq x_c \mid \mu)$$

$$= \int_{x_c}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2} \, dx$$

$$= 1 - \Phi\left(\frac{x_c - \mu}{\sigma}\right)$$

$$= \Phi\left(\frac{\mu - x_c}{\sigma}\right)$$



$$\Phi(-a) = 1 - \Phi(a)$$

$$= 1 - \Phi\left(\frac{\mu_0 + \sigma\Phi^{-1}(1-\alpha) - \mu}{\sigma}\right)$$

$$= \Phi\left(\frac{\mu - \mu_0}{\sigma} + \Phi^{-1}(\alpha)\right)$$

$\longrightarrow$ see plots on Week 4 extra slides.

Chain rule for pdfs

Recall conditional pdf

$$f(x_2, x_1) = f(x_2 \mid x_1)\, f(x_1)$$

$$f(x_n, \dots, x_1) = f(x_n \mid x_{n-1}, \dots x_1)\, \underbrace{f(x_{n-1}, \dots x_1)}$$

$$= f(x_{n-1} \mid x_{n-2} \dots x_1)$$

$$\vdots$$

$$\Rightarrow f(x_n, \dots x_1) = \prod_{i=1}^{n} f(x_i \mid \underbrace{x_{i-1}, \dots x_1})$$

if $i > 1$, else, no condition.

e.g. $f(x_3, x_2, x_1) = f(x_3 \mid x_2, x_1)$

$\times\ f(x_2 \mid x_1)$

$\times\ f(x_1)$.