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Discussion Session Week 5 (2 Nov 20)

1) $$f(x,y) = \begin{cases} 1 & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} z &= xy \\ u &= x \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= u \\ y &= z/u \end{aligned}$$

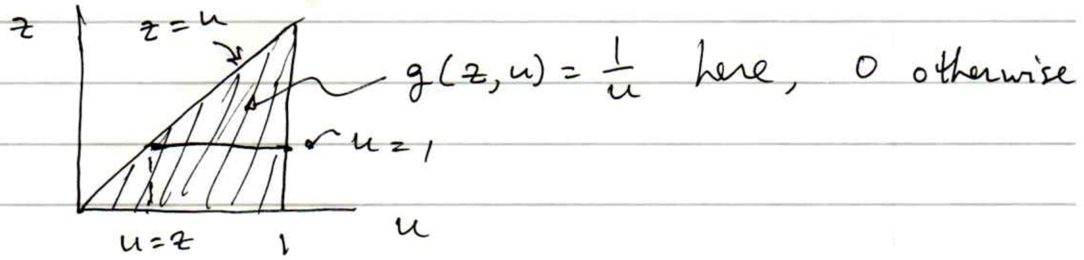
$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{u} & -\frac{z}{u^2} \end{vmatrix} = -\frac{1}{u}$$

$$g(z,u) = |J| \, f(x(z,u), y(z,u))$$

$$= \begin{cases} \frac{1}{u} & 0 \le u \le 1 \text{ and } 0 \le z \le u \\ 0 & \text{otherwise} \end{cases}$$

$0 \le x \le 1 \Rightarrow 0 \le u \le 1$

$0 \le y \le 1 \Rightarrow 0 \le xy \le x \Rightarrow$



$$g_z(z) = \int g(z,u)\, du = \int_z^1 \frac{du}{u} = -\ln z$$

$y \le 1 \Rightarrow xy \le x \Rightarrow z \le u$

1b) $g(z) = \int f(x, \frac{z}{x}) \frac{dx}{x}$

$$= \int_{x_{min}}^{x_{max}} 1 \cdot \frac{dx}{x}$$

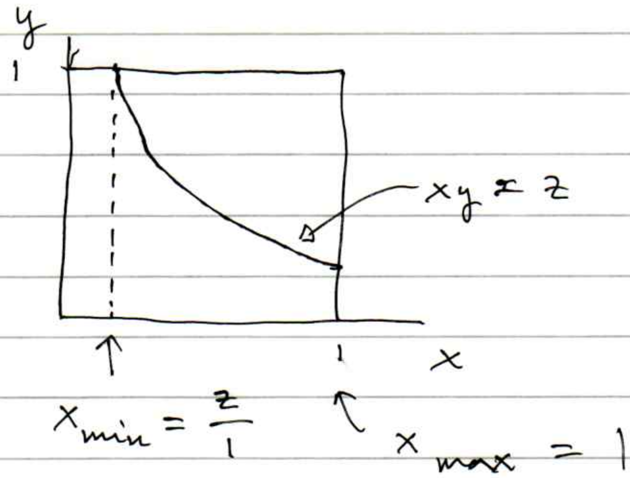
$f(x,y)$ is nonzero for

$0 \le x \le 1$

$0 \le y \le 1 \quad \Rightarrow \quad 0 \le \frac{z}{x} \le 1$

$\Rightarrow \quad 0 \le z \le x$

i.e. for a given value z



$$\Rightarrow g_z(z) = \int_z^1 \frac{dx}{x} = \ln x \Big|_z^1$$

$$= -\ln z$$

2) $$E[\alpha x + \beta] = \int (\alpha x + \beta) f(x)\, dx$$

$$= \alpha \int x f(x)\, dx + \beta$$

$$= \alpha E[x] + \beta$$

$$V[\alpha x + \beta] = E[(\alpha x + \beta)^2] - (E[\alpha x + \beta])^2$$

$$= E[\alpha^2 x^2 + 2\alpha\beta x + \beta^2] - (\alpha^2 \mu^2 + 2\alpha\beta\mu + \beta^2)$$

$\mu \equiv E[x]$

$$= \alpha^2 E[x^2] + \cancel{2\alpha\beta\mu} + \cancel{\beta^2} - \alpha^2\mu^2 - \cancel{2\alpha\beta\mu} - \cancel{\beta^2}$$

$$= \alpha^2 (E[x^2] - \mu^2)$$

$$= \alpha^2 V[x]$$

3a) $V[\alpha x + y] = E[(\alpha x + y)^2] - (E[\alpha x + y])^2$

$= E[\alpha^2 x^2 + 2\alpha xy + y^2] - (\alpha \mu_x + \mu_y)^2$

$= \alpha^2 E[x^2] + 2\alpha E[xy] + E[y^2]$
$\quad - \alpha^2 \mu_x^2 - 2\alpha \mu_x \mu_y - \mu_y^2$

$= \alpha^2 (E[x^2] - \mu_x^2) + E[y^2] - \mu_y^2$
$\quad + 2\alpha (E[xy] - \mu_x \mu_y)$

$= \alpha^2 V[x] + V[y] + 2\alpha \underbrace{\text{cov}[x,y]}_{= \rho \sigma_x \sigma_y}$

b) $V[\alpha x + y] = \alpha^2 \sigma_x^2 + \sigma_y^2 + 2\alpha \rho \sigma_x \sigma_y \geq 0$

Let $\alpha = \sigma_y / \sigma_x$

$\frac{\sigma_y^2}{\sigma_x^2} \sigma_x^2 + \sigma_y^2 + 2\left(\frac{\sigma_y}{\sigma_x}\right) \rho \sigma_x \sigma_y \geq 0$

$\Rightarrow 2\sigma_y^2 + 2\rho \sigma_y^2 \geq 0 \Rightarrow \underline{\underline{\rho \geq -1}}$

Let $\alpha = -\sigma_y / \sigma_x$

$\left(\frac{-\sigma_y}{\sigma_x}\right)^2 \sigma_x^2 + \sigma_y^2 - 2\frac{\sigma_y}{\sigma_x} \rho \sigma_x \sigma_y \geq 0$

$\sigma_y^2 + \sigma_y^2 - 2\rho \sigma_y^2 \geq 0 \Rightarrow \underline{\underline{\rho \leq +1}}$

4) $x_1 : \mu_1 = 10 , \sigma_1^2 = 1$

$x_2 : \mu_2 = 10 , \sigma_2^2 = 1$

$\mathrm{cov}[x_1 , x_2] = 0$

Let $y = \dfrac{x_1^2}{x_2}$

$$V[y] = \left(\frac{\partial y}{\partial x_1}\right)^2\Big|_{\vec{\mu}} \sigma_1^2 + \left(\frac{\partial y}{\partial x_2}\right)^2\Big|_{\vec{\mu}} \sigma_2^2$$

$$= \left(\frac{2x_1}{x_2}\right)^2\Big|_{\vec{\mu}} \sigma_1^2 + \left(-\frac{x_1^2}{x_2^2}\right)^2\Big|_{\vec{\mu}} \sigma_2^2$$

$$= 4 \cdot 1 + 1 \cdot 1 = 5$$

$$\Rightarrow \sigma_y = \sqrt{5} = 2.2$$

If $\mu_2 = 1$, then $y = x_1^2 / x_2$ is signficantly non linear in a region $x_2 \in \mu_2 \pm \sigma_2$ & therefore linear error prop. is a poor approx.