Week 9 Discussion Session (30 Nov 20) G. Cowan

Problem Sheet 6

1a) $n \sim \text{Poisson}(s+b)$

$$P(n|s,b) = \frac{(s+b)^n}{n!} e^{-(s+b)}$$

$b = 3.9$ (known)

Observe $n_{obs} = 16$

p-value of $s = 0$ $= P(n \geq n_{obs} | s=0, b=3.9)$

higher n more charac. of high $\underline{s}$.

$$= \sum_{n=n_{obs}}^{\infty} \frac{b^n}{n!} e^{-b}$$

$$= 1 - \sum_{n=0}^{n_{obs}-1} \frac{b^n}{n!} e^{-b}$$

$n_{dof} = 2(m+1)$

$m = n_{obs} - 1$

$$= F_{\chi^2}(2b; 2n_{obs})$$

$$= 1 - \text{TMath::Prob}(7.8, 32)$$

$$= 3.58 \times 10^{-6}$$

b) $Z = \Phi^{-1}(1-p) = 4.5$

$= \text{TMath::NormQuantile}(1-p)$

2a) $n_i \sim \text{Poisson}(\nu)$ $\quad i = 1, \ldots, N$ & indep.

$$L(\nu) = \prod_{i=1}^{N} \frac{\nu^{n_i} e^{-\nu}}{n_i!}$$

$$\ln L(\nu) = \sum_{i=1}^{N} \left(n_i \ln \nu - \nu \right) + C$$

$$\frac{\partial \ln L}{\partial \nu} = \sum_{i=1}^{N} \left(\frac{n_i}{\nu} - 1 \right) \overset{\text{set}}{=} 0$$

$$\Rightarrow \quad \hat{\nu} = \frac{1}{N} \sum_{i=1}^{N} n_i$$

b) $E[\hat{\nu}] = E\left[\frac{1}{N} \sum_{i=1}^{N} n_i \right] = \frac{1}{N} \sum_{i=1}^{N} \nu = \nu$

$$\Rightarrow \quad b = E[\hat{\nu}] - \nu = 0$$

$$V[\hat{\nu}] = V\left[\frac{1}{N} \sum_i n_i \right] = \frac{1}{N^2} \sum_i V[n_i]$$

(because all n_i indep.)

Also $V[n_i] = \nu$ since $n_i \sim$ Poisson

$$\Rightarrow V[\hat{\nu}] = \frac{\nu}{N} \quad \left[\text{or } \sigma_{\hat{\nu}} = \sqrt{\frac{\nu}{N}} \right]$$

c) $$\text{MVB} = - \frac{\left(1 + \frac{\partial b}{\partial \nu}\right)^2}{E\left[\frac{\partial^2 \ln L}{\partial \nu^2}\right]}$$

$$\frac{\partial^2 \ln L}{\partial \nu^2} = - \sum_{i=1}^{N} \frac{n_i}{\nu^2}$$

$$E\left[\frac{\partial^2 \ln L}{\partial \nu^2}\right] = - \frac{1}{\nu^2} \sum_{i=1}^{N} E[n_i] = - \frac{N}{\nu}$$

d) $$\text{MVB} = - \frac{(1+0)^2}{\left(-\frac{N}{\nu}\right)} = \frac{\nu}{N}$$

same as actual variance

$\Rightarrow$ $\hat{\nu}$ is efficient estimator for ν.

3) $$f(x) = \begin{cases} (1+\theta)\,x^{\theta} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

i.i.d sample $x_1, \ldots, x_n$

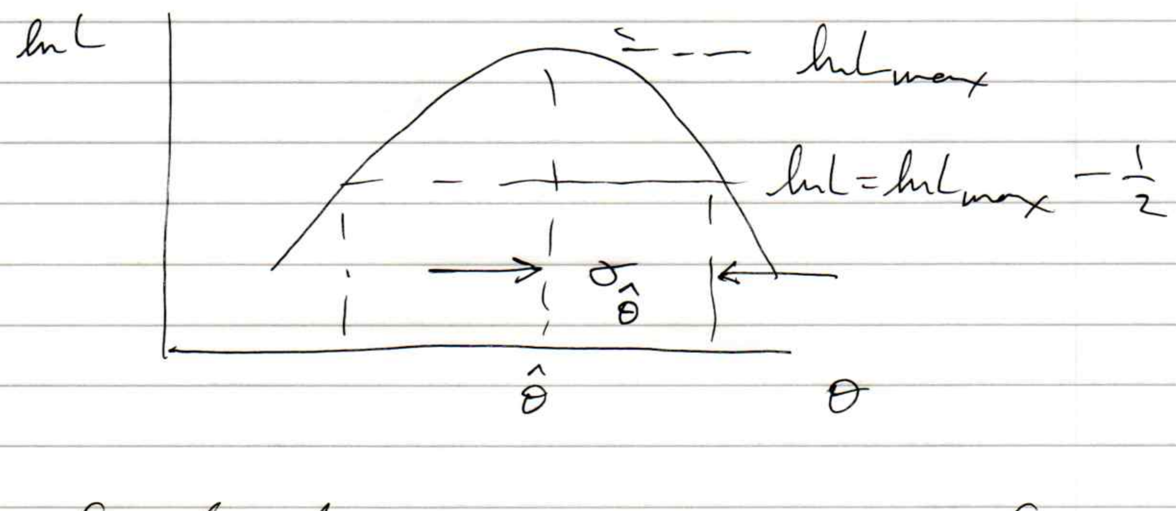
a) $$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} (1+\theta)\,x_i^{\theta}$$

$$\Rightarrow \ln L(\theta) = \sum_{i=1}^{n} \left[\ln(1+\theta) + \theta \ln x_i \right]$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{1+\theta} + \sum_{i=1}^{n} \ln x_i \overset{\text{set}}{=} 0$$

$$\Rightarrow \hat{\theta} = -1 - \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

b)



$\sigma_{\hat{\theta}}$ found by moving θ away from $\hat{\theta}$ until $\ln L$ decreases by $1/2$ from $\ln L_{max}$.

c) $\lambda = \exp\left[\frac{-1}{1+\theta}\right]$

Use $\hat{\lambda} = \lambda(\hat{\theta})$

$$\Rightarrow \hat{\lambda} = \exp\left[-\frac{1}{1+\left(-1-\frac{n}{\sum_i \ln x_i}\right)}\right]$$

$$= \exp\left[\frac{1}{n}\sum_i \ln x_i\right]$$

$$= \exp\left[\sum_i \ln x_i^{1/n}\right]$$

$$= \prod_{i=1}^{n} x_i^{1/n}$$

Or hard way: $\theta = -1 - \frac{1}{\ln\lambda}$

$$f(x;\lambda) = -\frac{1}{\ln\lambda}\, x^{\left(-1-\frac{1}{\ln\lambda}\right)}$$

$$L(\lambda) = \prod_{i=1}^{n} f(x_i;\lambda)$$

$$\rightarrow \ln L \longrightarrow \frac{\partial \ln L}{\partial\lambda} = 0 \longrightarrow \text{same } \hat{\lambda}$$

as above

Conf region from

$$\ln L(\vec{\theta}) = \ln L_{max} - \frac{1}{2} \underbrace{F^{-1}_{\chi^2_n}(1-\alpha)}_{Q_\alpha}$$

Cov. ellipse from

$$\ln L(\vec{\theta}) = \ln L_{max} - \frac{1}{2} \underbrace{(\vec{\theta}-\hat{\vec{\theta}})^T V^{-1} (\vec{\theta}-\hat{\vec{\theta}})}_{= s^2}$$

$\Rightarrow$ tangents at s std. deviations.

$\Rightarrow \quad Q_\alpha = s^2$ ← set in iminuit.

e.g. $n = 2$, $1-\alpha = 95\%$

$\Rightarrow \quad Q_\alpha = 5.99$