Discussion Session Notes Week 10

Problem Sheet 7

G. Cowan

1a) $x \sim \frac{x}{\theta^2} e^{-x/\theta}$ $\quad x \geq 0$

$\theta > 0$

[3] $E[x] = 2\theta$, $V[x] = 2\theta^2$

i.i.d. sample $x_1, \ldots, x_n$

$$L(\theta) = \prod_{i=1}^{n} \frac{x_i}{\theta} e^{-x_i/\theta}$$

$$\Rightarrow \ln L(\theta) = -2n \ln \theta - \sum_{i=1}^{n} \frac{x_i}{\theta} + C$$

$$\frac{\partial \ln L}{\partial \theta} = -\frac{2n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2} \overset{\text{set}}{=} 0$$

$$\Rightarrow \hat{\theta} = \frac{1}{2n} \sum_{i=1}^{n} x_i$$

1b) Use $E[x] = 2\theta$, $V[x] = 2\theta^2$

[4] $$\Rightarrow E[\hat{\theta}] = E\left[\frac{1}{2n} \sum_{i=1}^{n} x_i\right] = \frac{1}{2n} \sum_{i=1}^{n} E[x_i]$$

$$= \frac{n}{2n} \cdot 2\theta = \theta \Rightarrow b = 0$$

$$V[\hat{\theta}] = V\left[\frac{1}{2n} \sum_{i=1}^{n} x_i\right] = \frac{1}{4n^2} \sum_{i=1}^{n} V[x_i]$$

$$= \frac{1}{4n^2} \cdot n \cdot 2n^2 = \frac{\theta^2}{2n}$$

1(b) cont.

$$\mathrm{MVB} = -\frac{\left(1 + \cancel{\frac{\partial b}{\partial \theta}}^{\,0}\right)^2}{E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right]}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{2n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i$$

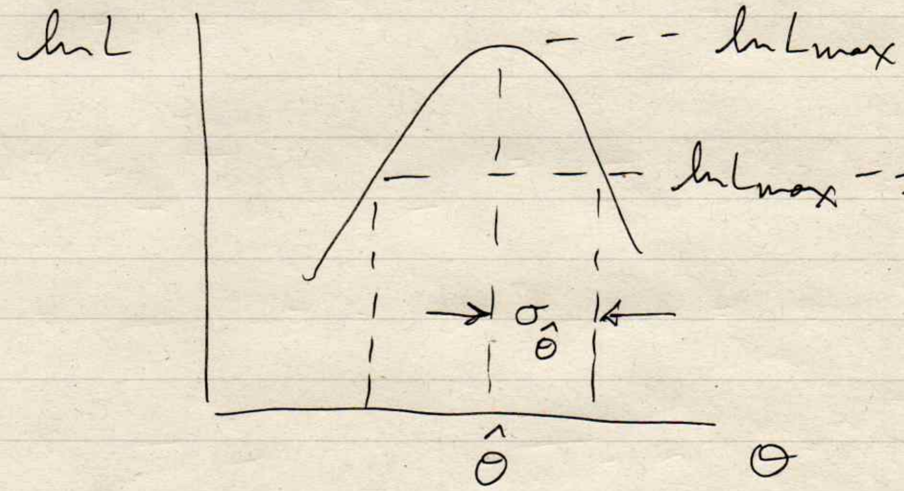
$$E\left[\ \ \right] = \frac{2n}{\theta^2} - \frac{2}{\theta^3}\underbrace{\sum_{i=1}^{n} \underbrace{E[x_i]}_{2\theta}}_{n \cdot 2\theta}$$

$$= -\frac{2n}{\theta^2}$$

$$\Rightarrow \mathrm{MVB} = \frac{\theta^2}{2n} \qquad \left(= V[\hat{\theta}] \right)$$

1c)

[2]



1d) $f(x;\theta) = \frac{x}{\theta^2} e^{-x/\theta}$, $E[x] = 2\theta$, $V[x] = 2\theta^2$

[4] $P(n;\theta) = \frac{(\alpha\theta^2)^n}{n!} e^{-\alpha\theta^2}$ Poisson, $E[n] = \nu$
$= \alpha\theta^2$
$\uparrow$ known

$$P(n, \vec{x}|\theta) = f(\vec{x}|n,\theta) \cdot P(n|\theta) = L(\theta)$$

$$= \frac{(\alpha\theta^2)^n}{n!} e^{-\alpha\theta^2} \prod_{i=1}^{n} \frac{x_i}{\theta^2} e^{-x_i/\theta}$$

$$\ln L(\theta) = \cancel{2n\ln\theta} - \alpha\theta^2 - \cancel{2n\ln\theta} + \sum_{i=1}^{n}\left(\ln x_i - \frac{x_i}{\theta}\right) + C$$

$$= -\alpha\theta^2 - \frac{1}{\theta}\sum_{i=1}^{n} x_i + C'$$

$$\frac{\partial \ln L}{\partial\theta} = -2\alpha\theta + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i \overset{\text{set}}{=} 0$$

$$\Rightarrow \hat{\theta} = \left(\frac{1}{2\alpha}\sum_{i=1}^{n} x_i\right)^{1/3}$$

1e) $$E[a(n,\vec{x})] = \sum_{n=0}^{\infty}\int a(\vec{x},n) \underbrace{P(n,\vec{x})}_{= f(\vec{x}|n)P(n)} d\vec{x}$$

[3]

$$= \sum_{n=0}^{\infty} P(n) \cdot \underbrace{\int a(\vec{x},n) f(\vec{x}|n)\, d\vec{x}}_{= E_{\vec{x}}[a(\vec{x},n)|n]}$$

$$= E_n\left[E_{\vec{x}}[a(\vec{x},n)|n]\right]$$

1f) $$\frac{\partial^2 \ln L}{\partial \theta^2} = -2\alpha - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i$$

depends on n & $\vec{x}$

$$\Rightarrow -E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = 2\alpha + \frac{2}{\theta^3} E\left[\sum_{i=1}^{n} x_i\right]$$

Use $E\left[\sum_{i=1}^{n} x_i\right] = E_n\left[\underbrace{E_{\vec{x}}\left[\sum_{i=1}^{n} x_i \,\middle|\, n\right]}_{n \cdot 2\theta}\right]$

$$= E_n\left[2n\theta\right] = 2 \cdot \underbrace{\alpha\theta^2}_{\nu} \cdot \theta = 2\alpha\theta^3$$

$$\Rightarrow -E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = 2\alpha + \frac{2}{\theta^3} \cdot 2\alpha\theta^3 = 6\alpha$$

$\Rightarrow V[\hat{\theta}] = \frac{1}{6\alpha}$ (assumes $b = 0$ & info ineq. $\rightarrow$ equality)

$E[n] \equiv \nu = \alpha\theta^2$

$\Rightarrow V[\hat{\theta}] = \frac{\theta^2}{6\nu}$ compare to case

where n fixed, $V[\hat{\theta}] = \frac{\theta^2}{2n}$

Now $V[\hat{\theta}]$ smaller because n provides additional info about θ.

x, y indep. 2014 Exam #2 1

$$\begin{cases} f(x \mid \theta_1, \theta_2) = \dfrac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\theta_1-\theta_2)^2/2\sigma^2} \\ g(y \mid \theta_2) = \dfrac{1}{\sqrt{2\pi}\,\sigma} e^{-(y-\theta_2)^2/2\sigma^2} \end{cases}$$

a) $L(\theta_1, \theta_2) = P(x, y \mid \theta_1, \theta_2) = f(x \mid \theta_1, \theta_2)\, g(y \mid \theta_2)$

$$= \frac{1}{2\pi\sigma^2} \exp\left[- \frac{(x-\theta_1-\theta_2)^2 + (y-\theta_2)^2}{2\sigma^2} \right]$$

$$\ln L(\theta_1, \theta_2) = -\frac{1}{2} \left[\frac{(x-\theta_1-\theta_2)^2 + (y-\theta_2)^2}{\sigma^2} \right]$$

b) $$\frac{\partial \ln L}{\partial \theta_1} = \frac{x - \theta_1 - \theta_2}{\sigma^2} \overset{\text{set}}{=} 0$$

$$\frac{\partial \ln L}{\partial \theta_2} = \underbrace{\frac{x - \theta_1 - \theta_2}{\sigma^2}}_{\text{cancel}} + \frac{y - \theta_2}{\sigma^2} \overset{\text{set}}{=} 0$$

$$\Rightarrow \boxed{\begin{aligned} \theta_2 &= y \\ \theta_1 &= x - y \end{aligned}}$$

c) $E[\hat{\theta}_1] = E[x] - E[y] = \theta_1 + \theta_2 - \theta_2 = \theta_1$ ✓

$E[\hat{\theta}_2] = E[y] = \theta_2$

$V[\hat{\theta}_1] = V[x] + V[y] = 2\sigma^2$

$V[\hat{\theta}_2] = V[y] = \sigma^2$

c (cont.)

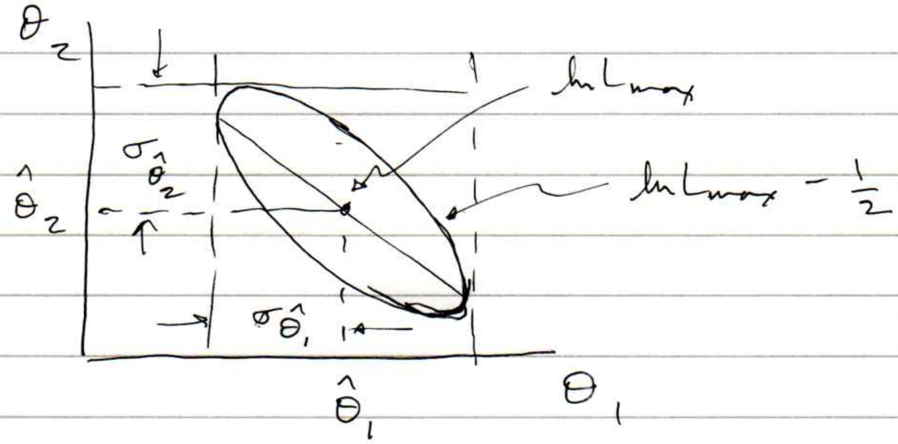
$$\mathrm{cov}[\hat{\theta}_1, \hat{\theta}_2] = \mathrm{cov}[x-y, y]$$

$$= \underbrace{\cancel{\mathrm{cov}[x,y]}}_{0} - \mathrm{cov}[y,y] \qquad \text{(show)}$$

$$= -\sigma^2$$

$$\Rightarrow \rho_{12} = -\frac{\sigma^2}{\sqrt{2}\,\sigma \cdot \sigma} = -\frac{1}{\sqrt{2}}$$

d)



e) $\hat{\hat{\theta}}_2(\theta_1)$ from sol'n to $\frac{\partial \ln L}{\partial \theta_2} = 0$

$$\frac{x - \theta_1 - \theta_2}{\cancel{\sigma^2}} + \frac{y - \theta_2}{\cancel{\sigma^2}} = 0$$

$$\hat{\hat{\theta}}_2(\theta_1) = \frac{1}{2}(x + y - \theta_1)$$

$$\Rightarrow \ln L_p(\theta_1) = -\frac{1}{2}\left[\frac{(x - \theta_1 - \frac{1}{2}(x+y-\theta_1))^2 + (y - \frac{1}{2}(x+y-\theta_1))^2}{\sigma^2}\right]$$

$$\cancel{= -\frac{1}{2\sigma^2}\left((x-\theta_1)^2 - (x-\theta_1)(x+y-\theta_1) + \frac{1}{4}(x+y-\theta_1)^2 + y^2 - 2\theta_1 y + \theta_1^2\right)}$$

$$\ln L_p(\theta_1) = -\frac{1}{2\sigma^2}\left[\left(\frac{x}{2} - \frac{y}{2} - \frac{\theta_1}{2}\right)^2 + \left(\frac{y}{2} - \frac{x}{2} + \frac{\theta_1}{2}\right)^2\right]$$

$$= -\frac{1}{4}\frac{(x - y - \theta_1)^2}{\sigma^2}$$

$$\frac{\partial^2 \ln L_p}{\partial \theta_1^2} = -\frac{1}{2\sigma^2}$$

$$\Rightarrow -E\left[\frac{\partial^2 \ln L}{\partial \theta_1^2}\right] = \frac{1}{2\sigma^2}$$

$$\Rightarrow V[\hat{\theta}_1] \approx I^{-1}(\theta) = 2\sigma^2$$ ($\leftarrow$ same as before.)