Statistical Data Analysis 2020/21
Lecture Week 9

London Postgraduate Lectures on Particle Physics

University of London MSc/MSci course PH4515

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Statistical Data Analysis
Lecture 9-1

- Least squares with histogram data

LS with histogram data

The fit function in an LS fit is not a pdf, but it could be proportional to one, e.g., when we fit the “envelope” of a histogram.

Suppose for example, we have an i.i.d. data sample of n values $x_1,\ldots, x_n$ sampled from a pdf $f(x;\boldsymbol{\theta})$. Goal is to estimate $\boldsymbol{\theta}$.

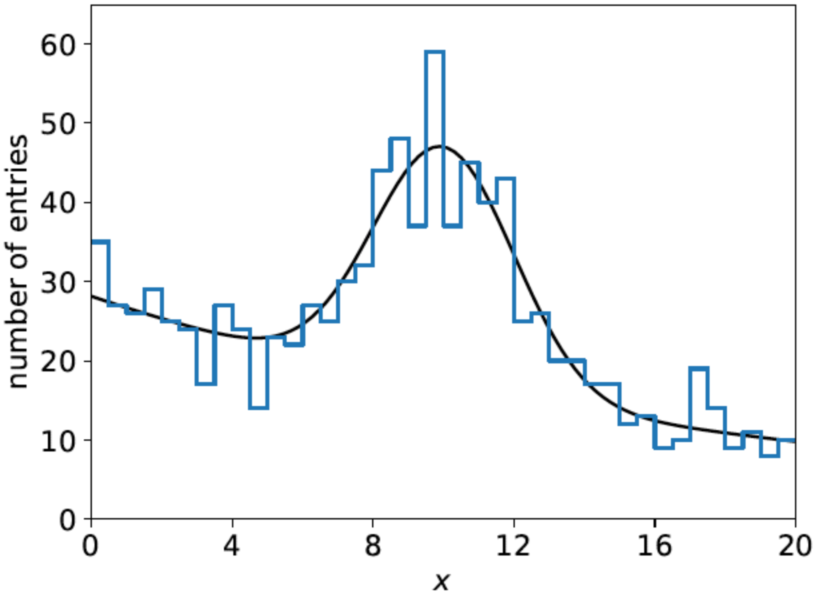
Instead of using all n values, put them in a histogram with N bins, i.e., y_i = number of entries in bin i: $\boldsymbol{y} = (y_1,\ldots, y_N)$.

The model predicts mean values:

$$
\begin{aligned}
E[y_i] &= \mu_i(\boldsymbol{\theta}) \\
&= n \int_{\mathrm{bin}\, i} f(x;\boldsymbol{\theta})\, dx \\
&\approx n f(x_i;\boldsymbol{\theta})\, \Delta x
\end{aligned}
$$

bin centre

bin width



LS with histogram data (2)

The usual models:

for fixed sample size n, take $\boldsymbol{y} \sim$ multinomial,
if n not fixed, $y_i \sim$ Poisson(μ_i)

Suppose that the expected number of entries in each μ_i are all $\gg$ 1 and probability to be in any individual bin $p_i \ll 1$, one can show

$\rightarrow y_i$ indep. and $\sim$ Gauss with $\sigma_i \approx \surd \mu_i$. ($\rightarrow \sigma_i$ depends on $\boldsymbol{\theta}$).

The (log-) likelihood functions are then

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_i(\boldsymbol{\theta})} e^{-(y_i - \mu_i(\boldsymbol{\theta}))^2/2\sigma_i^2(\boldsymbol{\theta})}$$

$$\ln L(\boldsymbol{\theta}) = -\frac{1}{2}\sum_{i=1}^{N} \frac{(y_i - \mu_i(\boldsymbol{\theta}))^2}{\sigma_i(\boldsymbol{\theta})^2} - \sum_{i=1}^{N} \ln \sigma_i(\boldsymbol{\theta}) + C$$

LS with histogram data (3)

Still define the least-squares estimators to minimize

$$\chi^2(\boldsymbol{\theta}) = \sum_{i=1}^{N} \frac{(y_i - \mu_i(\boldsymbol{\theta}))^2}{\sigma_i(\boldsymbol{\theta})^2}$$

No longer equivalent to maximum likelihood (equal for $\mu_i \gg 1$).

Two possibilities for σ_i:

$\sigma_i = \sqrt{\mu_i(\boldsymbol{\theta})}$ (LS method)

$\sigma_i = \sqrt{y_i}$ (Modified LS method)

Modified LS can be easier computationally but not defined if any $y_i = 0$.

For either method, $\chi^2_{\min} \sim$ chi-square pdf for $\mu_i \gg 1$, but this breaks down for when the μ_i are not large.

LS with histogram data — normalization

Do not "fit" the normalization, i.e., $n \longrightarrow$ free parameter ν:

$$\mu_i(\boldsymbol{\theta}, \nu) = \nu \int_{\mathrm{bin}\, i} f(x; \boldsymbol{\theta})\, dx$$

If you do this, one finds the LS estimator for ν is not n, but rather

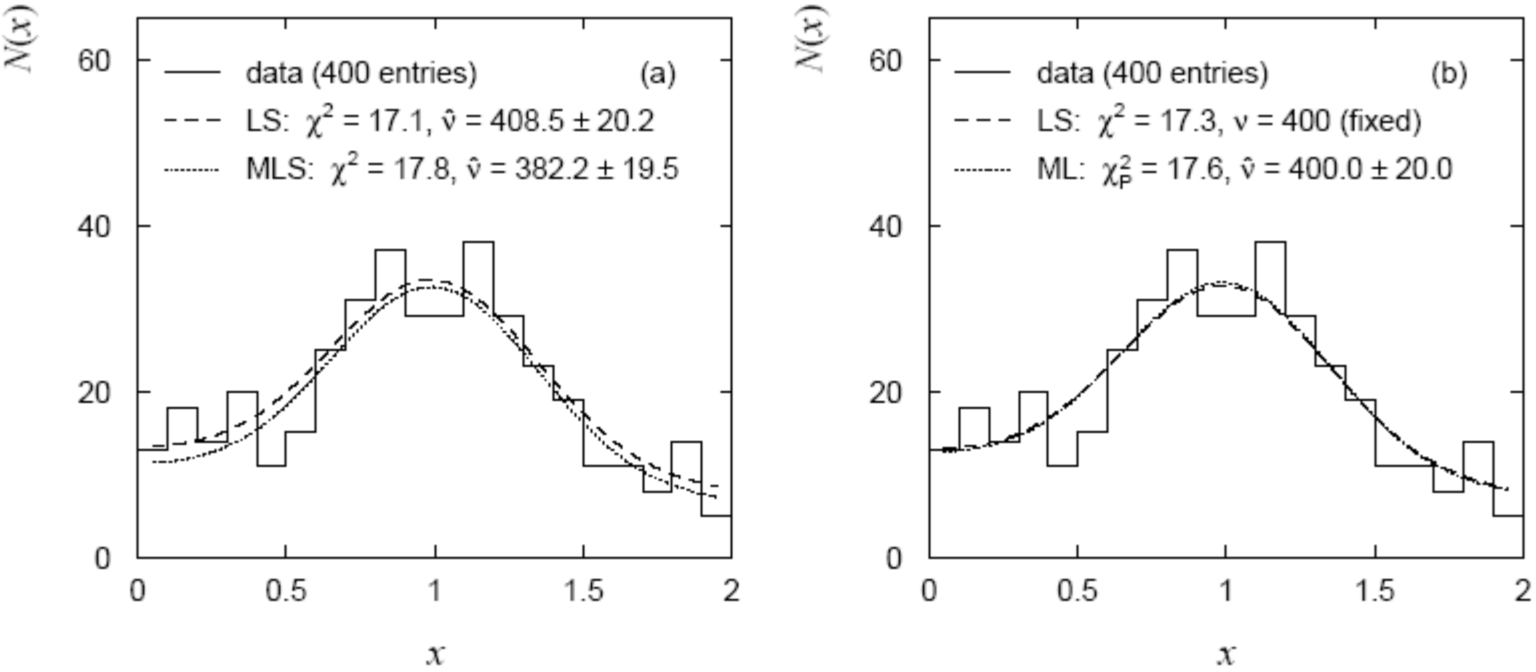
$$\hat{\nu}_{\mathrm{LS}} = n + \frac{\chi^2_{\mathrm{min}}}{2}$$

$$\hat{\nu}_{\mathrm{MLS}} = n - \chi^2_{\mathrm{min}}$$

Software may include adjustable normalization parameter as default; better to use known n.

LS normalization example

Example with $n = 400$ entries, $N = 20$ bins:



Expect $\chi^2_{\min}$ around $N - m$,

→ relative error in $\hat{\nu}$ large when N large, n small

Either get n directly from data for LS (or better, use ML).

Statistical Data Analysis
Lecture 9-2

- Goodness-of-fit from the likelihood ratio
- Wilks' theorem
- MLE and goodness-of-fit all in one

Goodness of fit from the likelihood ratio

Suppose we model data using a likelihood $L(\boldsymbol{\mu})$ that depends on N parameters $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)$. Define the statistic

$$t_{\boldsymbol{\mu}} = -2 \ln \frac{L(\boldsymbol{\mu})}{L(\hat{\boldsymbol{\mu}})}$$

where $\hat{\boldsymbol{\mu}}$ is the ML estimator for $\boldsymbol{\mu}$. Value of $t_{\boldsymbol{\mu}}$ reflects agreement between hypothesized $\boldsymbol{\mu}$ and the data.

Good agreement means $\boldsymbol{\mu} \approx \hat{\boldsymbol{\mu}}$, so $t_{\boldsymbol{\mu}}$ is small;

Larger $t_{\boldsymbol{\mu}}$ means less compatibility between data and $\boldsymbol{\mu}$.

Quantify "goodness of fit" with p-value: $p_{\boldsymbol{\mu}} = \int_{t_{\boldsymbol{\mu},\mathrm{obs}}}^{\infty} \underset{\nearrow\ \text{need this pdf}}{f(t_{\boldsymbol{\mu}}|\boldsymbol{\mu})}\, dt_{\boldsymbol{\mu}}$

Likelihood ratio (2)

Now suppose the parameters $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)$ can be determined by another set of parameters $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_M)$, with $M < N$.

Want to test hypothesis that the true model is somewhere in the subspace $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\theta})$ versus the alternative of the full parameter space $\boldsymbol{\mu}$. Generalize the LR test statistic to be

fit M parameters

$$t_{\boldsymbol{\mu}} = -2 \ln \frac{L(\boldsymbol{\mu}(\hat{\boldsymbol{\theta}}))}{L(\hat{\boldsymbol{\mu}})}$$

fit N parameters

To get p-value, need pdf $f(t_{\mu}|\boldsymbol{\mu}(\boldsymbol{\theta}))$.

Wilks' Theorem

Wilks' Theorem: if the hypothesized $\mu_i(\boldsymbol{\theta})$, $i = 1,\dots,N$, are true for some choice of the parameters $\boldsymbol{\theta} = (\theta_1,\dots, \theta_M)$, then in the large sample limit (and provided regularity conditions are satisfied)

MLE of $(\theta_1,\dots, \theta_M)$

$$t_{\boldsymbol{\mu}} = -2 \ln \frac{L(\boldsymbol{\mu}(\hat{\boldsymbol{\theta}}))}{L(\hat{\boldsymbol{\mu}})}$$

follows a chi-square distribution for $N - M$ degrees of freedom.

MLE of $(\mu_1,\dots, \mu_N)$

The regularity conditions include: the model in the numerator of the likelihood ratio is "nested" within the one in the denominator, i.e., $\boldsymbol{\mu}(\boldsymbol{\theta})$ is a special case of $\boldsymbol{\mu} = (\mu_1,\dots, \mu_N)$.

Proof boils down to having all estimators ~ Gaussian.

S.S. Wilks, *The large-sample distribution of the likelihood ratio for testing composite hypotheses*, Ann. Math. Statist. **9** (1938) 60-2.

Goodness of fit with Gaussian data

Suppose the data are N independent Gaussian distributed values:

$$y_i \sim \text{Gauss}(\mu_i, \sigma_i) \ , \qquad i = 1, \ldots, N$$

want to estimate

known

N measurements and N parameters (= “saturated model”)

Likelihood:

$$L(\boldsymbol{\mu}) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma_i} e^{-(y_i - \mu_i)^2/2\sigma_i^2}$$

Log-likelihood:

$$\ln L(\boldsymbol{\mu}) = -\frac{1}{2} \sum_{i=1}^{N} \frac{(y_i - \mu_i)^2}{\sigma_i^2} + C$$

ML estimators:

$$\hat{\mu}_i = y_i \qquad i = 1, \ldots, N$$

Likelihood ratio for Gaussian data

Now suppose $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\theta})$, e.g., in an LS fit with $\mu_i(\boldsymbol{\theta}) = \mu(x_i; \boldsymbol{\theta})$.

The goodness-of-fit statistic for the test of the hypothesis $\boldsymbol{\mu}(\boldsymbol{\theta})$ becomes

$$t_{\boldsymbol{\mu}} = -2 \ln \frac{L(\boldsymbol{\mu}(\hat{\boldsymbol{\theta}}))}{L(\hat{\boldsymbol{\mu}})} = \sum_{i=1}^{N} \frac{(y_i - \mu_i(\hat{\boldsymbol{\theta}}))^2}{\sigma_i^2} \sim \chi^2_{N-M}$$

chi-square pdf for N-M
degrees of freedom

Here $t_{\boldsymbol{\mu}}$ is the same as χ^2_{min} from an LS fit.

So Wilks' theorem formally states the property that we claimed for the minimized chi-squared from an LS fit with N measurements and M fitted parameters.

Likelihood ratio for Poisson data

Suppose the data are a set of values $\boldsymbol{n} = (n_1, \ldots, n_N)$, e.g., the numbers of events in a histogram with N bins.

Assume $n_i \sim$ Poisson(ν_i), $i = 1, \ldots, N$, all independent.

First (for LR denominator) treat $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_N)$ as all adjustable:

Likelihood:
$$L(\boldsymbol{\nu}) = \prod_{i=1}^{N} \frac{\nu_i^{n_i}}{n_i!} e^{-\nu_i}$$

Log-likelihood:
$$\ln L(\boldsymbol{\nu}) = \sum_{i=1}^{N} [n_i \ln \nu_i - \nu_i] + C$$

ML estimators:
$$\hat{\nu}_i = n_i \;, \qquad i = 1, \ldots, N$$

Goodness of fit with Poisson data (2)

For LR numerator find $\boldsymbol{\nu}(\boldsymbol{\theta})$ with M fitted parameters $\boldsymbol{\theta} = (\theta_1,\ldots, \theta_M)$:

$$t_{\boldsymbol{\nu}} = -2 \ln \frac{L(\boldsymbol{\nu}(\hat{\boldsymbol{\theta}}))}{L(\hat{\boldsymbol{\nu}})} = -2 \sum_{i=1}^{N} \left[n_i \ln \frac{\nu_i(\hat{\boldsymbol{\theta}})}{n_i} - \nu_i(\hat{\boldsymbol{\theta}}) + n_i \right]$$

Wilks' theorem: in large-sample limit $\quad t_{\boldsymbol{\nu}} \sim \chi^2_{N-M}$

Exact in large sample limit; in practice good approximation for surprisingly small n_i (~several).

As before use t_ν to get p-value of $\boldsymbol{\nu}(\boldsymbol{\theta})$,

independent of $\boldsymbol{\theta}$

$$p_{\boldsymbol{\nu}} = \int_{t_{\boldsymbol{\nu}},\mathrm{obs}}^{\infty} f(t_{\boldsymbol{\nu}} | \boldsymbol{\nu}(\boldsymbol{\theta}))\, dt_{\boldsymbol{\nu}} = 1 - F_{\chi^2}(t_{\boldsymbol{\nu},\mathrm{obs}}; N - M)$$

Goodness of fit with multinomial data

Similar if data $\boldsymbol{n} = (n_1, \ldots, n_N)$ follow multinomial distribution:

$$P(\mathbf{n}|\mathbf{p}, n_{\rm tot}) = \frac{n_{\rm tot}!}{n_1! n_2! \ldots n_N!} p_1^{n_1} p_2^{n_2} \ldots p_N^{n_N}$$

E.g. histogram with N bins but fix: $n_{\rm tot} = \sum_{i=1}^{N} n_i$

Log-likelihood: $\ln L(\boldsymbol{\nu}) = \sum_{i=1}^{N} n_i \ln \frac{\nu_i}{n_{\rm tot}} + C$ $\quad (\nu_i = p_i n_{\rm tot})$

ML estimators: $\hat{\nu}_i = n_i$ (Only N–1 independent; one is $n_{\rm tot}$ minus sum of rest.)

Goodness of fit with multinomial data (2)

The likelihood ratio statistics become:

$$t_{\boldsymbol{\nu}} = -2 \ln \frac{L(\boldsymbol{\nu}(\hat{\boldsymbol{\theta}}))}{L(\hat{\boldsymbol{\nu}})} = -2 \sum_{i=1}^{N} n_i \ln \frac{\nu_i(\hat{\boldsymbol{\theta}})}{n_i}$$

Wilks: in large sample limit $t_{\boldsymbol{\nu}} \sim \chi^2_{N-M-1}$

One less degree of freedom than in Poisson case because effectively only N–1 parameters fitted in denominator of LR.

Estimators and g.o.f. all at once

Evaluate numerators with $\boldsymbol{\theta}$ (not its estimator); if any $n_i = 0$, omit the corresponding log terms:

$$\chi^2_{\rm P}(\boldsymbol{\theta}) \;=\; -2\sum_{i=1}^{N}\left[n_i \ln \frac{\nu_i(\boldsymbol{\theta})}{n_i} - \nu_i(\boldsymbol{\theta}) + n_i\right] \qquad \text{(Poisson)}$$

$$\chi^2_{\rm M}(\boldsymbol{\theta}) \;=\; -2\sum_{i=1}^{N} n_i \ln \frac{\nu_i(\boldsymbol{\theta})}{n_i} \qquad \text{(Multinomial)}$$

These are equal to the corresponding $-2 \ln L(\boldsymbol{\theta})$ plus terms not depending on $\boldsymbol{\theta}$, so minimizing them gives the usual ML estimators for $\boldsymbol{\theta}$.

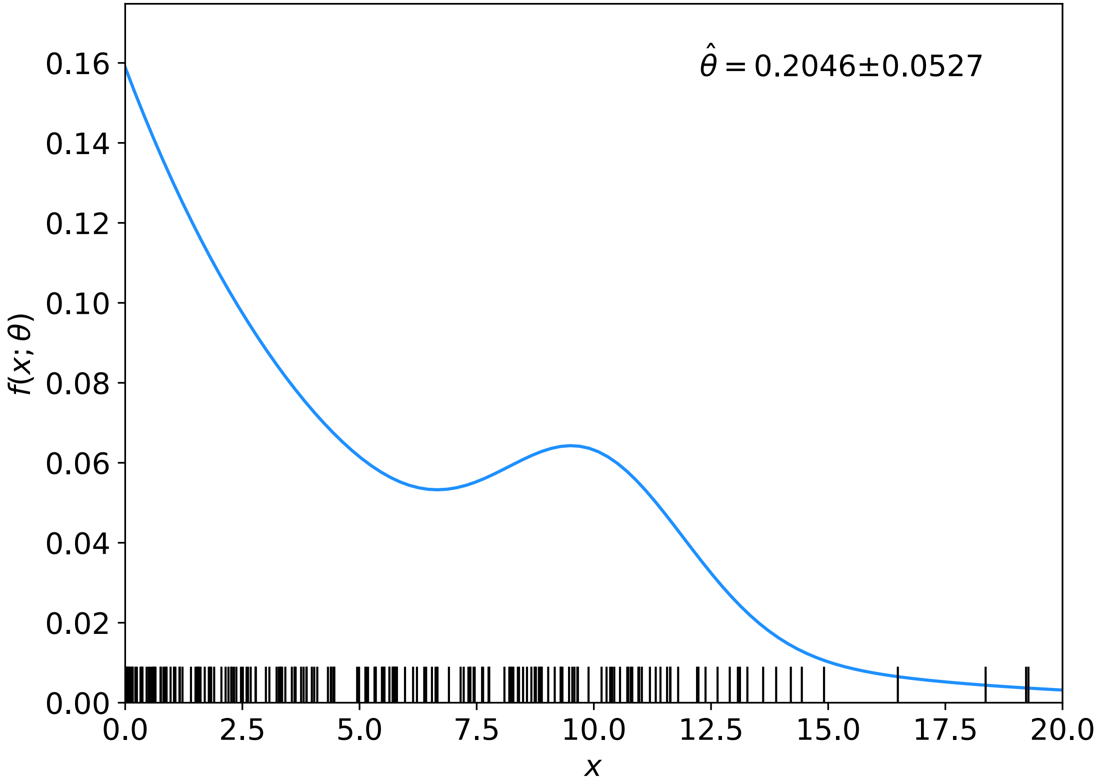
The minimized value gives the statistic t_ν, so we get goodness-of-fit for free.

Steve Baker and Robert D. Cousins, *Clarification of the use of the chi-square and likelihood functions in fits to histograms*, NIM **221** (1984) 437.

Examples of ML/LS fits

Unbinned maximum likelihood (mlFit.py, minimize negLogL)

$$\ln L(\boldsymbol{\theta}) = \sum_{i=1}^{n} \ln f(x_i; \boldsymbol{\theta})$$

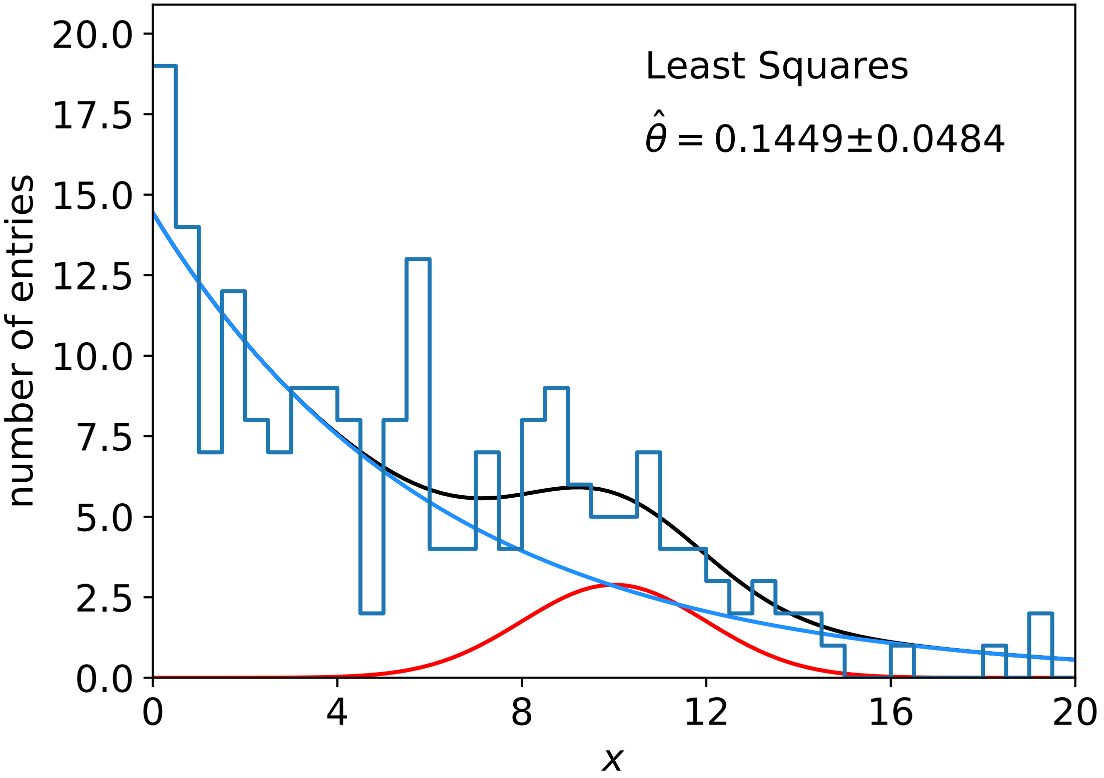


No useful measure of goodness-of-fit from unbinned ML.

Examples of ML/LS fits

Least Squares fit (histFit.py, minimize chi2LS)

$$\chi^2(\boldsymbol{\theta}) = \sum_{i=1}^{N} \frac{(y_i - \mu_i(\boldsymbol{\theta}))^2}{\mu_i(\boldsymbol{\theta})}$$



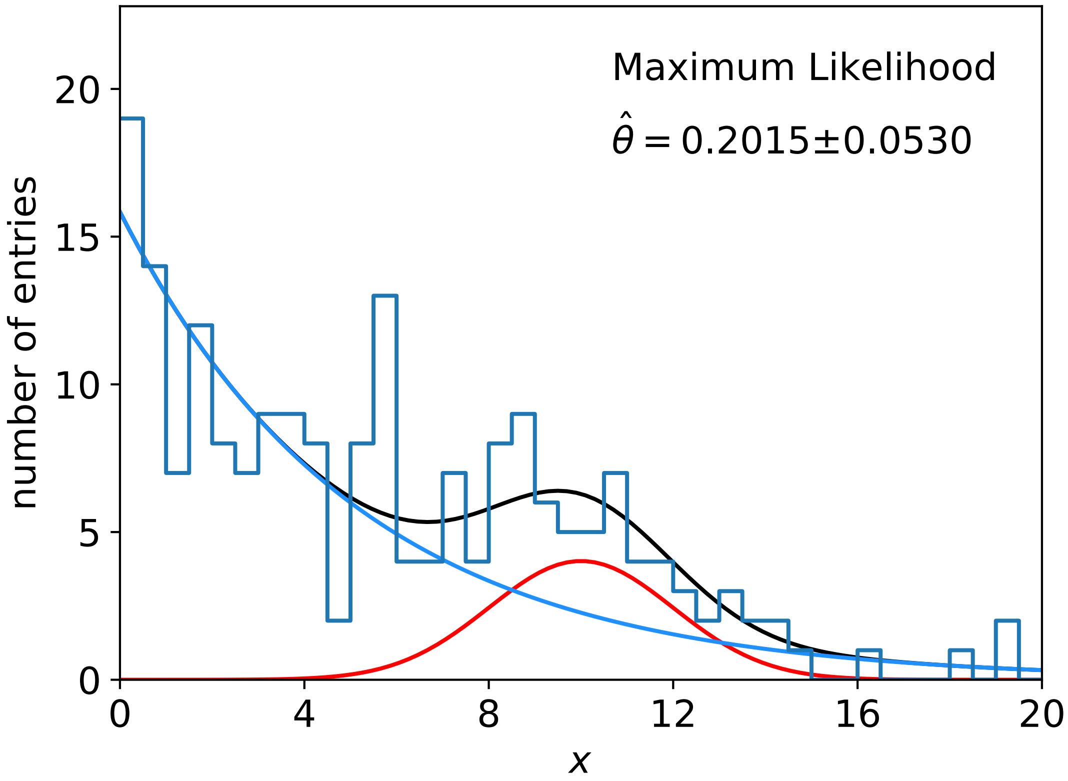
$$\chi^2_{\min} = 32.7$$
$$n_{\mathrm{dof}} = 38$$
$$p = 0.71$$

Many bins with few entries, LS not expected to be reliable.

Examples of ML/LS fits

Multinomial maximum likelihood fit (histFit.py, minimize chi2M)

$$\chi^2_{\mathrm{M}}(\boldsymbol{\theta}) = -2\sum_{i=1}^{N} n_i \ln \frac{\nu_i(\boldsymbol{\theta})}{n_i}$$



$\chi^2_{\mathrm{min}} = 35.3$

$n_{\mathrm{dof}} = 37$

$p = 0.55$

Essentially same result as unbinned ML.

Statistical Data Analysis
Lecture 9-3

- Interval estimation
- Confidence interval from inverting a test
- Example: limits on mean of Gaussian

Confidence intervals by inverting a test

In addition to a ‘point estimate’ of a parameter we should report an interval reflecting its statistical uncertainty.

Confidence intervals for a parameter θ can be found by defining a test of the hypothesized value θ (do this for all θ):

Specify values of the data that are ‘disfavoured’ by θ (critical region) such that $P(\text{data in critical region} \mid \theta) \leq \alpha$ for a prespecified α, e.g., 0.05 or 0.1.

If data observed in the critical region, reject the value θ.

Now invert the test to define a confidence interval as:

set of θ values that are not rejected in a test of size α (confidence level CL is $1-\alpha$).

Relation between confidence interval and p-value

Equivalently we can consider a significance test for each hypothesized value of θ, resulting in a p-value, p_θ.

If $p_\theta \leq \alpha$, then we reject θ.

The confidence interval at CL $= 1 - \alpha$ consists of those values of θ that are not rejected.

E.g. an upper limit on θ is the greatest value for which $p_\theta > \alpha$.

In practice find by setting $p_\theta = \alpha$ and solve for θ.

For a multidimensional parameter space $\boldsymbol{\theta} = (\theta_1, \ldots\ \theta_M)$ *use* same idea – result is a confidence "region" with boundary determined by $p_{\boldsymbol{\theta}} = \alpha$.

Coverage probability of confidence interval

If the true value of θ is rejected, then it's not in the confidence interval. The probability for this is by construction (equality for continuous data):

$$P(\text{reject } \theta | \theta) \leq \alpha = \text{type-I error rate}$$

Therefore, the probability for the interval to contain or "cover" θ is

$$P(\text{conf. interval “covers” } \theta | \theta) \geq 1 - \alpha$$

This assumes that the set of θ values considered includes the true value, i.e., it assumes the composite hypothesis $P(\boldsymbol{x}|H,\theta)$.

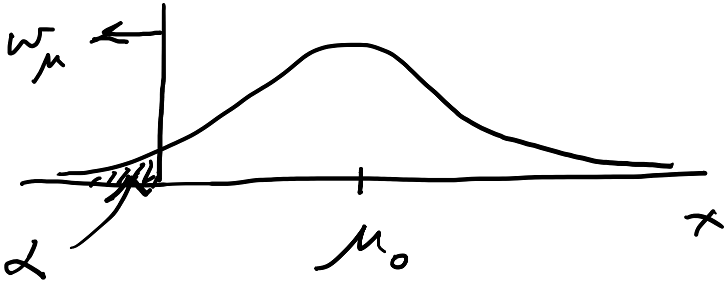
Example: upper limit on mean of Gaussian

When we test the parameter, we should take the critical region to maximize the power with respect to the relevant alternative(s).

Example: $x \sim$ Gauss(μ, σ) (take σ known)

Test $H_0 : \mu = \mu_0$ versus the alternative $H_1 : \mu < \mu_0$

→ Put w_μ at region of x-space characteristic of low μ (i.e. at low x)

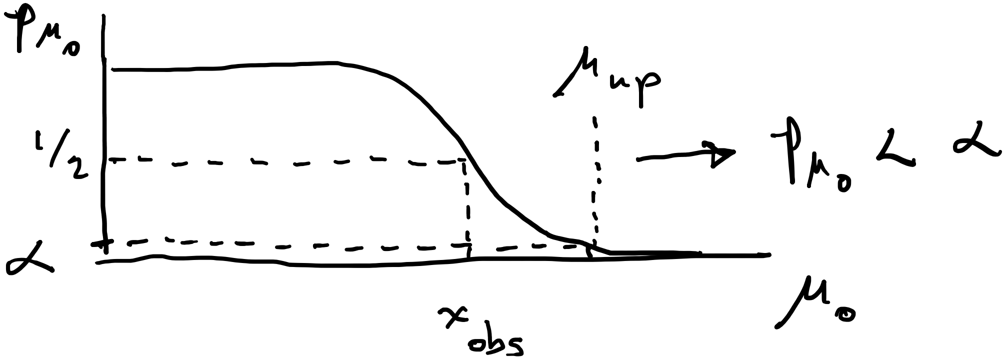


Equivalently, take the p-value to be

$$p_{\mu_0} = P(x \leq x_{\rm obs}|\mu_0) = \int_{-\infty}^{x_{\rm obs}} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu_0)^2/2\sigma^2}\, dx = \Phi\left(\frac{x_{\rm obs}-\mu_0}{\sigma}\right)$$

Upper limit on Gaussian mean (2)

To find confidence interval, repeat for all μ_0, i.e., set $p_{\mu 0} = \alpha$ and solve for μ_0 to find the interval's boundary



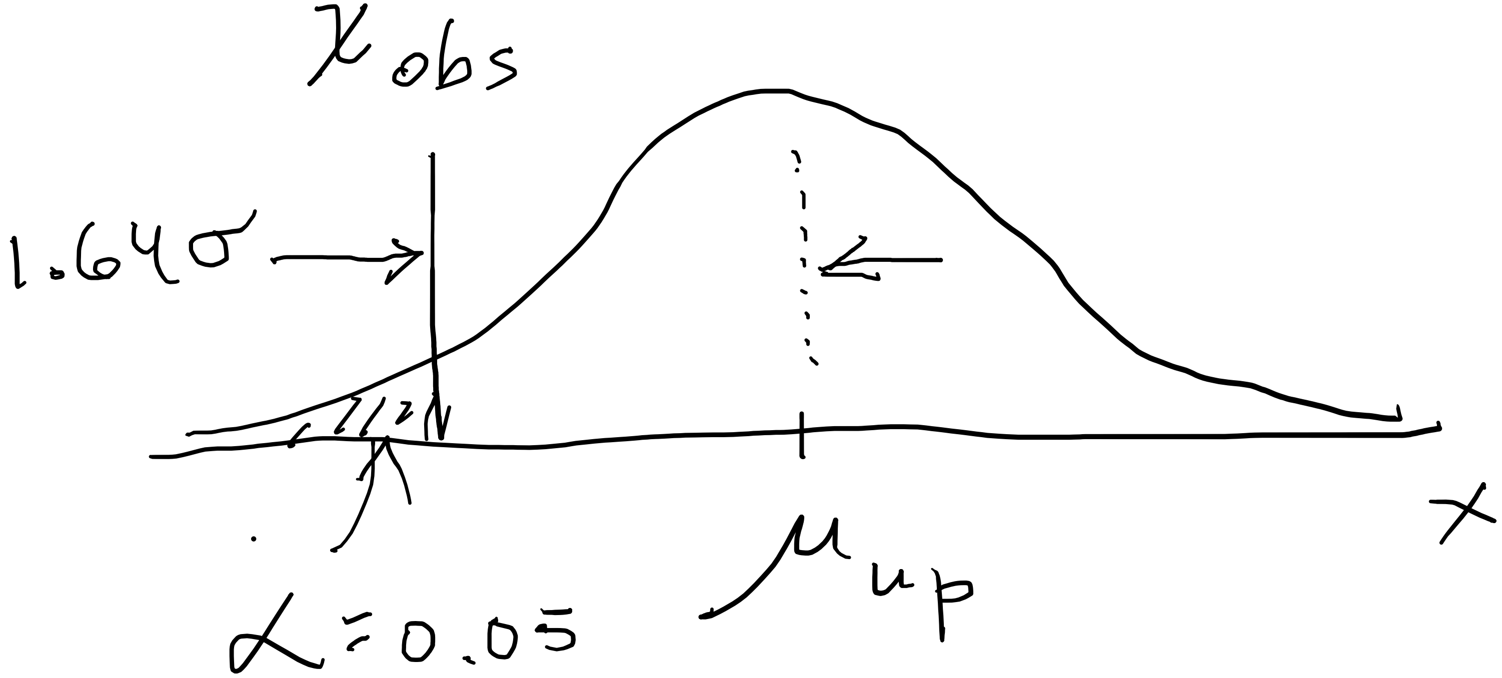
$$\mu_0 \to \mu_{\rm up} = x_{\rm obs} - \sigma \Phi^{-1}(\alpha) = x_{\rm obs} + \sigma \Phi^{-1}(1-\alpha)$$

This is an upper limit on μ, i.e., higher μ have even lower p-value and are in even worse agreement with the data.

Usually use $\Phi^{-1}(\alpha) = -\Phi^{-1}(1-\alpha)$ so as to express the upper limit as $x_{\rm obs}$ plus a positive quantity. E.g. for α = 0.05, $\Phi^{-1}(1-0.05)$ = 1.64.

Upper limit on Gaussian mean (3)

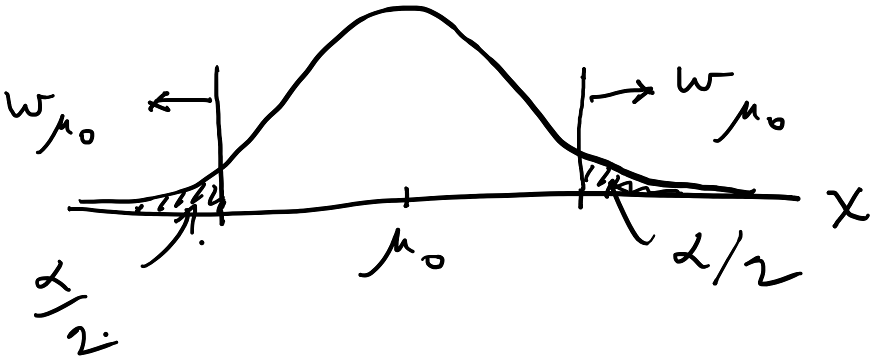
μ_{up} = the hypothetical value of μ such that there is only a probability α to find $x < x_{\mathrm{obs}}$.



1- vs. 2-sided intervals

Now test: $H_0 : \mu = \mu_0$ versus the alternative $H_1 : \mu \neq \mu_0$

I.e. we consider the alternative to μ_0 to include higher and lower values, so take critical region on both sides:



Result is a “central” confidence interval $[\mu_{\rm lo}, \mu_{\rm up}]$:

$$\mu_{\rm lo} = x_{\rm obs} - \sigma \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$$

$$\mu_{\rm up} = x_{\rm obs} + \sigma \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$$

E.g. for $\alpha = 0.05$

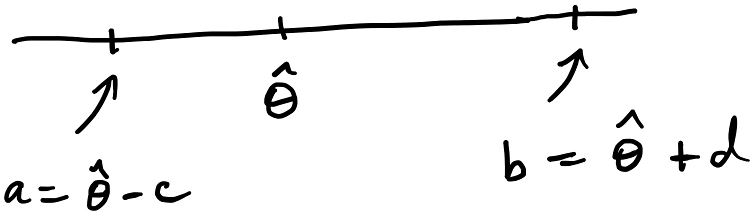
$$\Phi^{-1}\left(1 - \frac{\alpha}{2}\right) = 1.96 \approx 2$$

Note upper edge of two-sided interval is higher (i.e. not as tight of a limit) than obtained from the one-sided test.

On the meaning of a confidence interval

Often we report the confidence interval $[a,b]$ together with the point estimate as an “asymmetric error bar”, e.g.,

$\hat{\theta}^{+d}_{-c}$



E.g. (at CL = 1 – α = 68.3%): $\hat{\theta} = 80.25^{+0.31}_{-0.25}$

Does this mean P(80.00 < θ < 80.56) = 68.3%? No, not for a frequentist confidence interval. The parameter θ does not fluctuate upon repetition of the measurement; the endpoints of the interval do, i.e., the endpoints of the interval fluctuate (they are functions of data):

$$P(a(x) < \theta < b(x)) = 1 - \alpha$$

Statistical Data Analysis
Lecture 9-4

- Confidence intervals from the likelihood function

Approximate confidence intervals/regions from the likelihood function

Suppose we test parameter value(s) $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ using the ratio

$$\lambda(\boldsymbol{\theta}) = \frac{L(\boldsymbol{\theta})}{L(\hat{\boldsymbol{\theta}})} \qquad 0 \leq \lambda(\boldsymbol{\theta}) \leq 1$$

Lower $\lambda(\boldsymbol{\theta})$ means worse agreement between data and hypothesized $\boldsymbol{\theta}$. Equivalently, usually define

$$t_{\boldsymbol{\theta}} = -2 \ln \lambda(\boldsymbol{\theta})$$

so higher t_θ means worse agreement between $\boldsymbol{\theta}$ and the data.

p-value of $\boldsymbol{\theta}$ therefore

$$p_{\boldsymbol{\theta}} = \int_{t_{\boldsymbol{\theta},\mathrm{obs}}}^{\infty} \underbrace{f(t_{\boldsymbol{\theta}}|\boldsymbol{\theta})}_{\text{need pdf}} \, dt_{\boldsymbol{\theta}}$$

Confidence region from Wilks' theorem

Wilks' theorem says (in large-sample limit and provided certain conditions hold...)

$$f(t_{\boldsymbol{\theta}}|\boldsymbol{\theta}) \sim \chi^2_n$$

chi-square dist. with # d.o.f. = # of components in $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$.

Assuming this holds, the p-value is

$$p_{\boldsymbol{\theta}} = 1 - F_{\chi^2_n}(t_{\boldsymbol{\theta}})$$

← set equal to α

To find boundary of confidence region set $p_\theta = \alpha$ and solve for t_θ:

$$t_{\boldsymbol{\theta}} = F^{-1}_{\chi^2_n}(1-\alpha)$$

Recall also

$$t_{\boldsymbol{\theta}} = -2\ln\frac{L(\boldsymbol{\theta})}{L(\hat{\boldsymbol{\theta}})}$$

Confidence region from Wilks' theorem (cont.)

i.e., boundary of confidence region in $\boldsymbol{\theta}$ space is where

$$\ln L(\boldsymbol{\theta}) = \ln L(\hat{\boldsymbol{\theta}}) - \tfrac{1}{2} F^{-1}_{\chi^2_n}(1-\alpha)$$

For example, for $1 - \alpha = 68.3\%$ and $n = 1$ parameter,

$$F^{-1}_{\chi^2_1}(0.683) = 1$$

and so the 68.3% confidence level interval is determined by

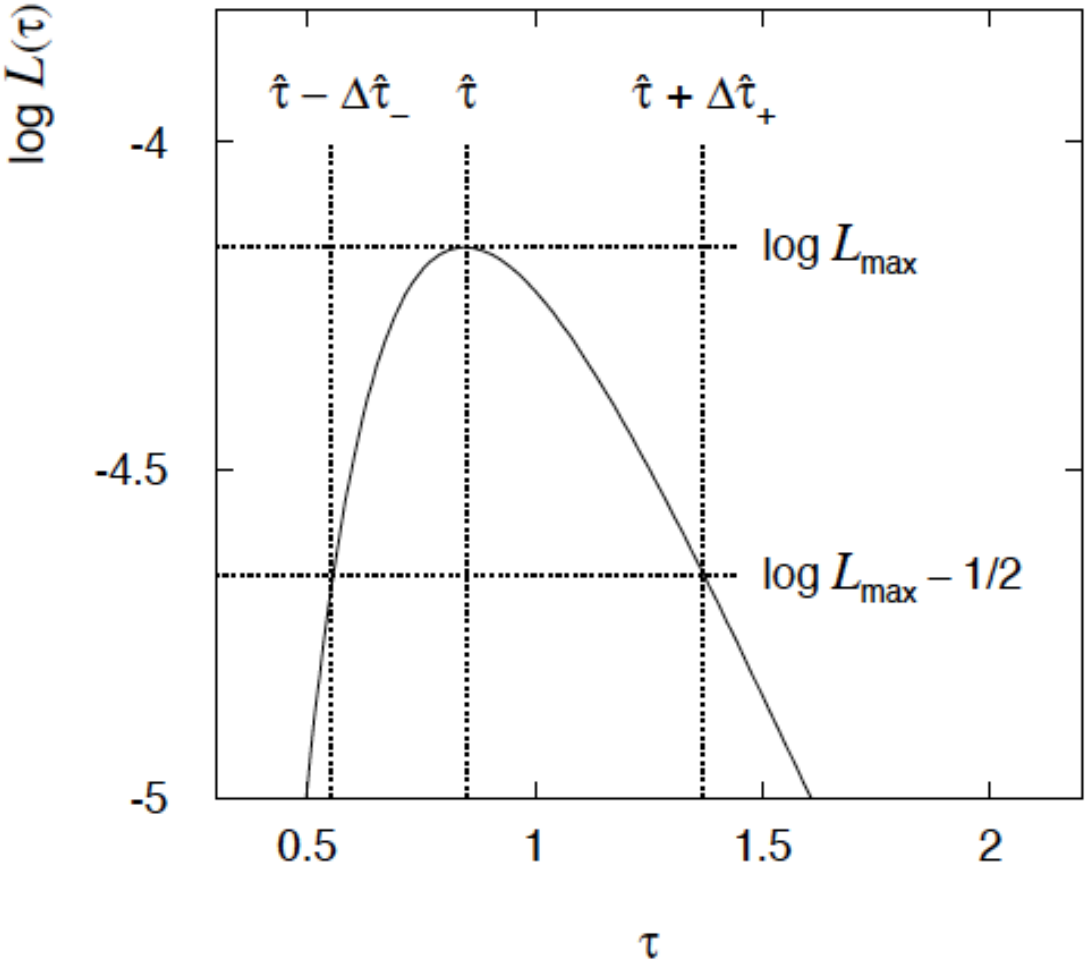
$$\ln L(\theta) = \ln L(\hat{\theta}) - \frac{1}{2}$$

Same as recipe for finding the estimator's standard deviation, i.e.,

$[\hat{\theta} - \sigma_{\hat{\theta}}, \hat{\theta} + \sigma_{\hat{\theta}}]$ is a 68.3% CL confidence interval.

Example of interval from $\ln L(\theta)$

For $n=1$ parameter, CL = 0.683, $Q_\alpha = 1$.



Our exponential example, now with only $n = 5$ events.

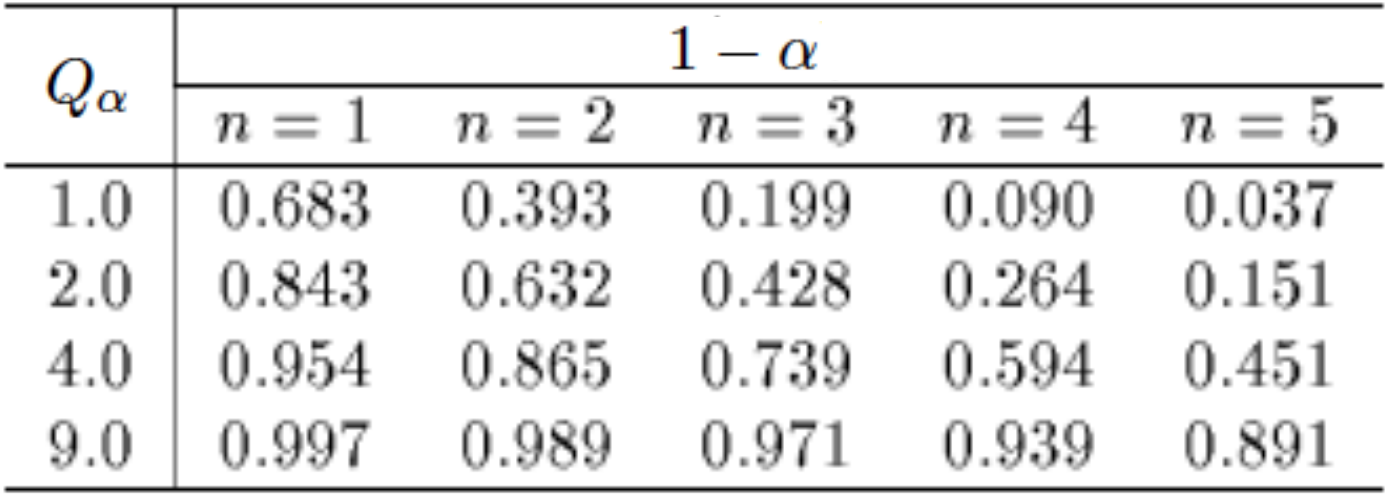
Can report ML estimate with approx. confidence interval from $\ln L_{\max} - 1/2$ as "asymmetric error bar":

$$\hat{\tau} = 0.85^{+0.52}_{-0.30}$$

Multiparameter case

For increasing number of parameters, CL $= 1 - \alpha$ decreases for confidence region determined by a given

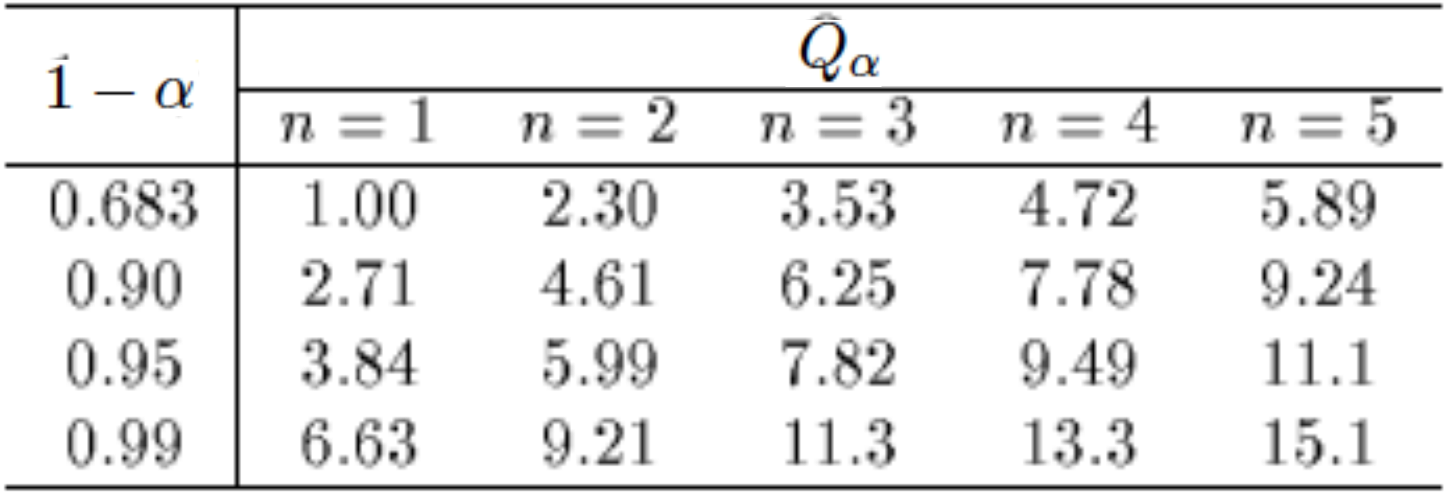
$$Q_\alpha = F^{-1}_{\chi^2_n}(1 - \alpha)$$



Q_α	$1-\alpha$				
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
1.0	0.683	0.393	0.199	0.090	0.037
2.0	0.843	0.632	0.428	0.264	0.151
4.0	0.954	0.865	0.739	0.594	0.451
9.0	0.997	0.989	0.971	0.939	0.891

Multiparameter case (cont.)

Equivalently, Q_α increases with n for a given CL $= 1 - \alpha$.

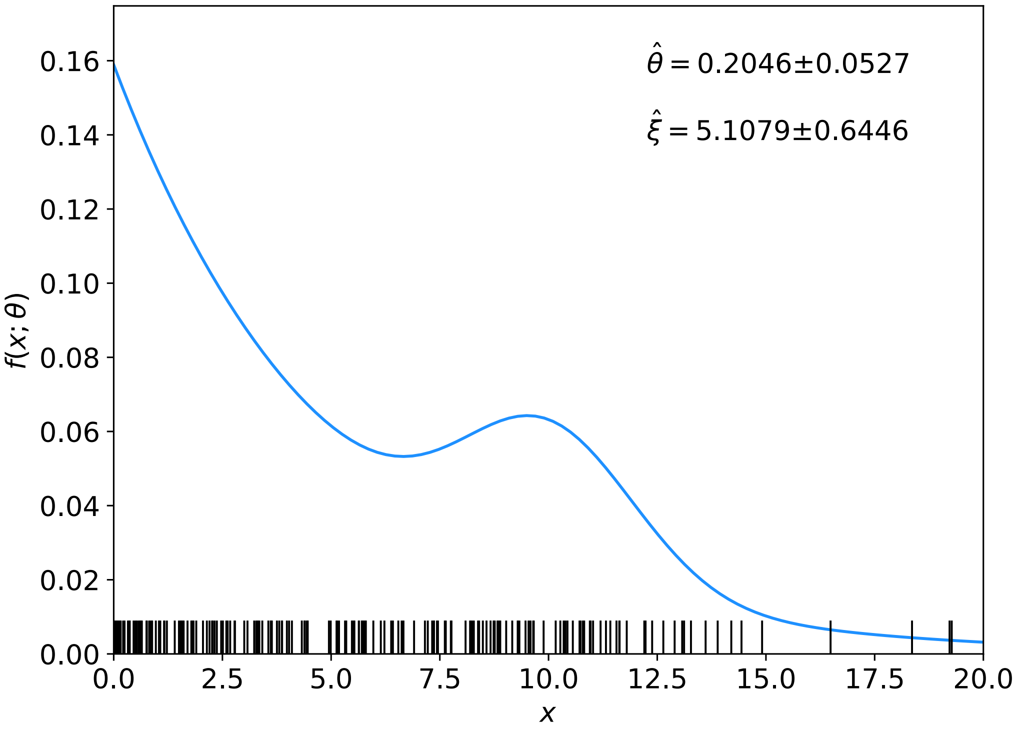


$1-\alpha$	Q_α				
	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
0.683	1.00	2.30	3.53	4.72	5.89
0.90	2.71	4.61	6.25	7.78	9.24
0.95	3.84	5.99	7.82	9.49	11.1
0.99	6.63	9.21	11.3	13.3	15.1

Example: 2 parameter fit:

Example from problem sheet 8, i.i.d. sample of size 200

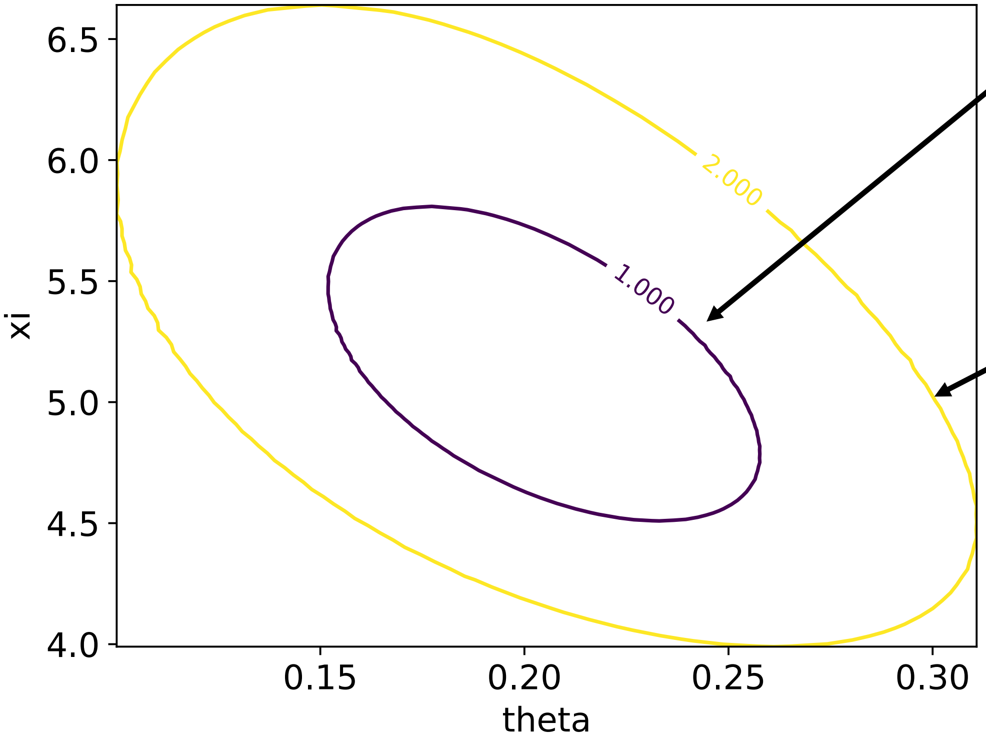
$$x \sim f(x;\theta,\xi) = \theta \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} + (1-\theta)\frac{1}{\xi}e^{-x/\xi}$$



Here fit two parameters: θ and ξ.

Example: 2 parameter fit:

In iminuit, user can set nsigma $= \sqrt{Q_\alpha}$



nsigma=1

Q_α = 1

CL = 0.393

nsigma=2

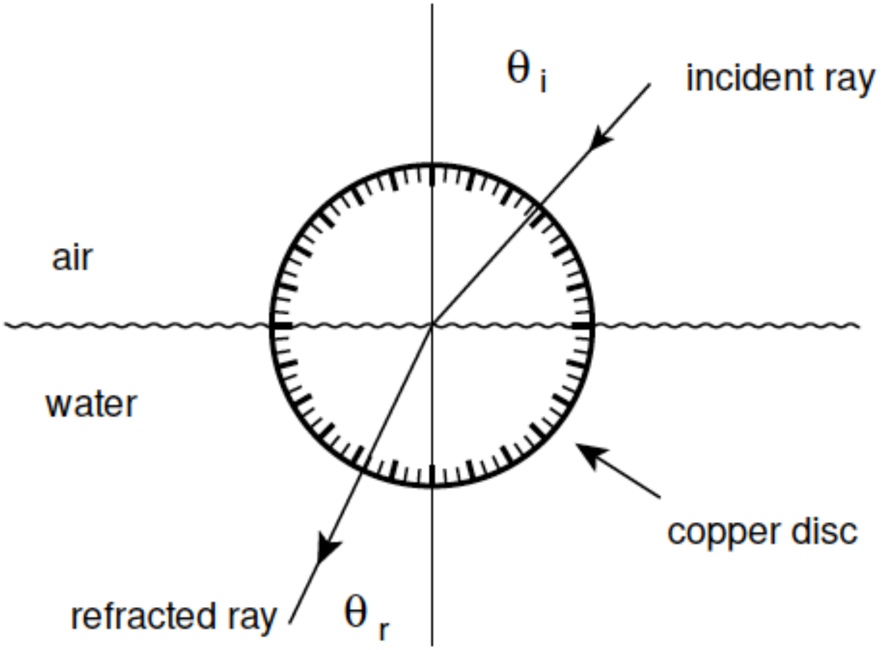
Q_α = 4

CL = 0.865

Extra slides

LS example: refraction data from Ptolemy

Astronomer Claudius Ptolemy obtained data on refraction of light by water in around 140 A.D.:



Angles of incidence and refraction (degrees)

θ_i	θ_r
10	8
20	$15\frac{1}{2}$
30	$22\frac{1}{2}$
40	29
50	35
60	$40\frac{1}{2}$
70	$45\frac{1}{2}$
80	50

Suppose the angle of incidence is set with negligible error, and the measured angle of refraction has a standard deviation of ½°   .

Laws of refraction

A commonly used law of refraction was

$$\theta_{\rm r} = \alpha \theta_{\rm i} \ ,$$

although it is reported that Ptolemy preferred

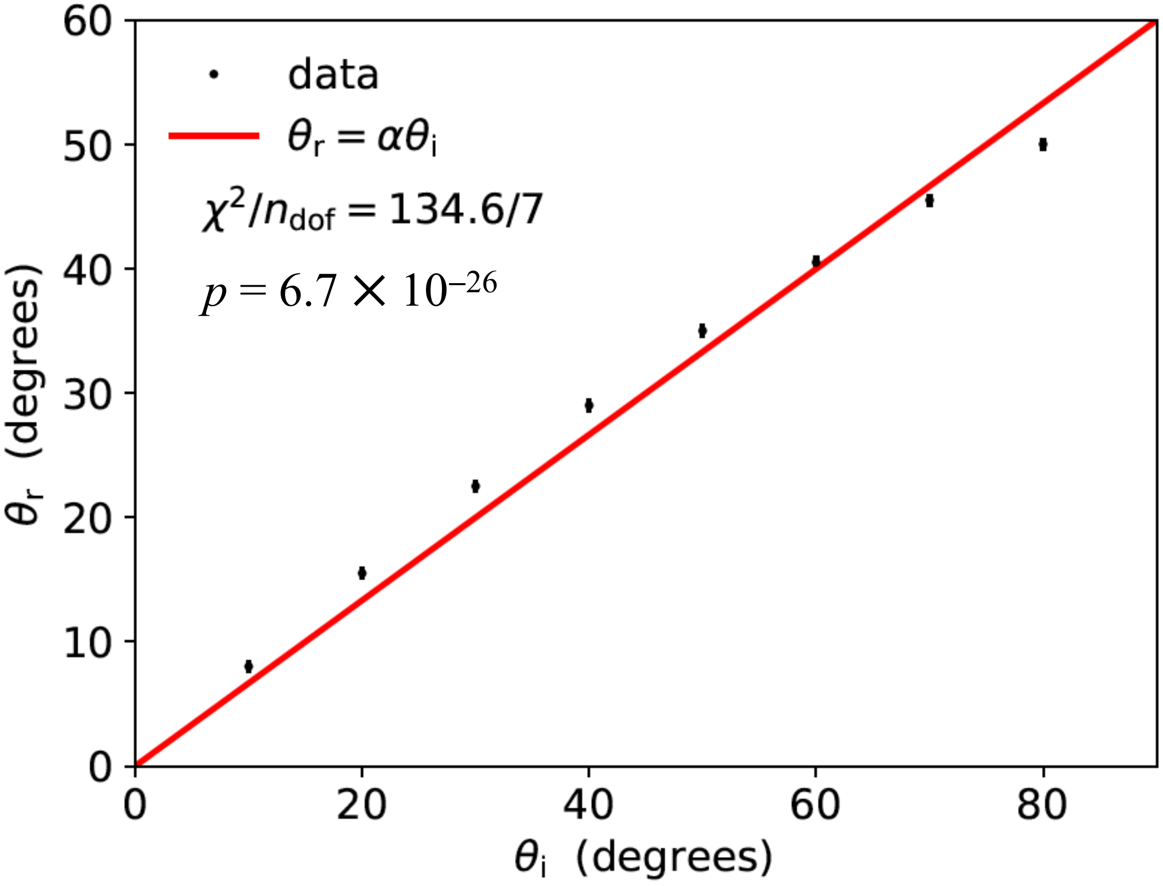
$$\theta_{\rm r} = \alpha \theta_{\rm i} - \beta \theta_{\rm i}^2 \ .$$

The law of refraction discovered by Ibn Sahl in 984 (and rediscovered by Snell in 1621) is

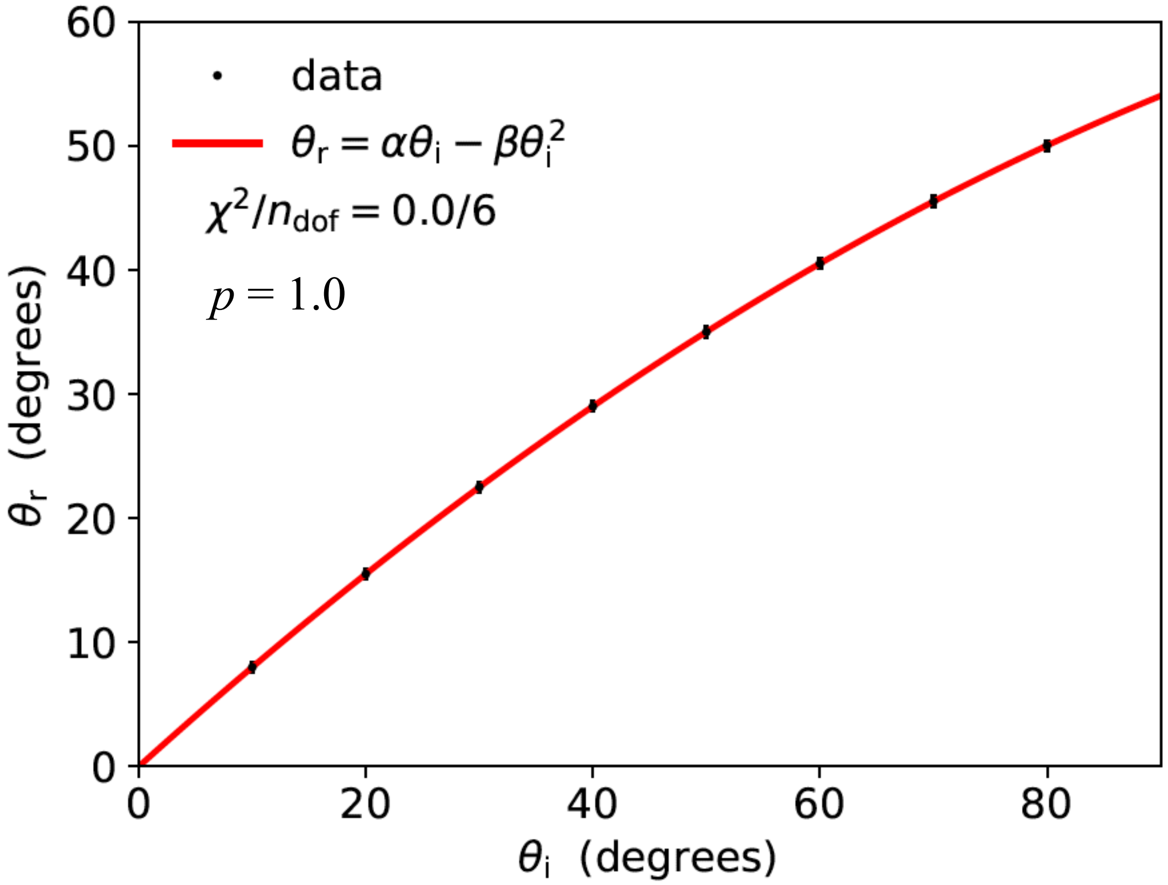
$$\theta_{\rm r} = \sin^{-1} \left(\frac{\sin \theta_{\rm i}}{r} \right) .$$

where $r = n_{\rm r}/n_{\rm i}$ is the ratio of indices of refraction of the two media.

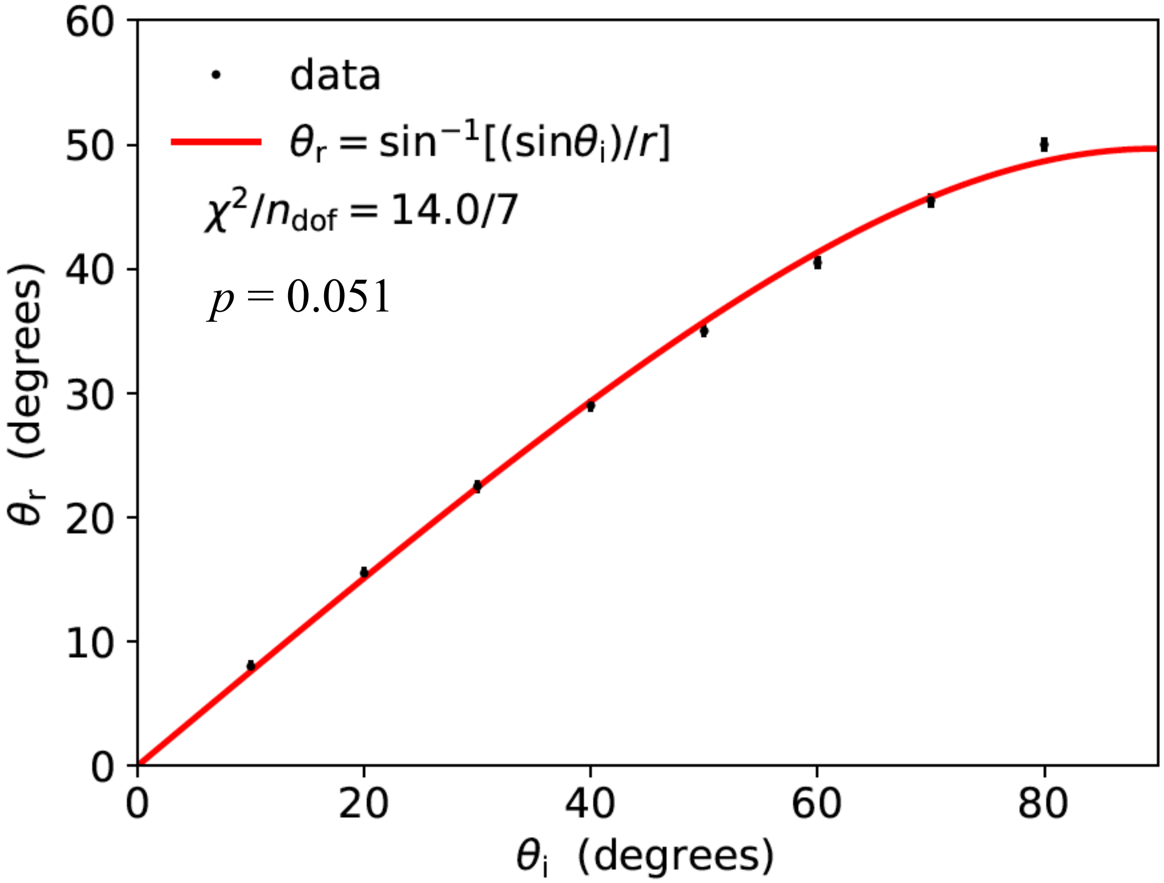
LS fit: $\theta_r = \alpha\theta_i$



LS fit: $\theta_\mathrm{r} = \alpha\theta_\mathrm{i} - \beta\theta_\mathrm{i}^2$



LS fit: Snell's Law



Fitted index of refraction of water r = 1.3116 ± 0.0056 found not quite compatible with currently known value 1.330.