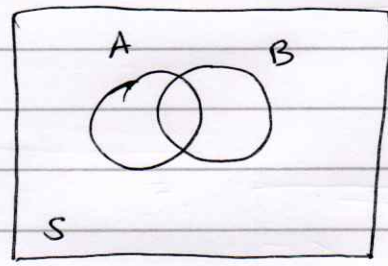
Discussion Session Week 1

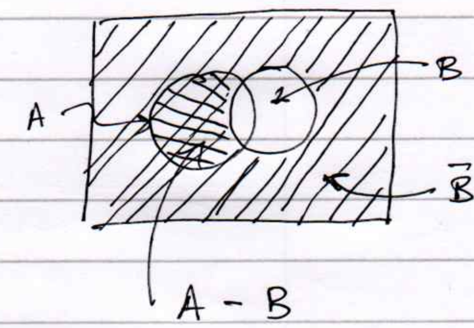
1) Example w/ Kolmogorov axioms:

Show $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Recall

$A - B \equiv A \cap \bar{B}$



$A \cup B = (A - A \cap B) \cup B$

disjoint

$\Rightarrow P(A \cup B) = P(A - A \cap B) + P(B)$ ①

also $A = (A - A \cap B) \cup A \cap B$ (disjoint)

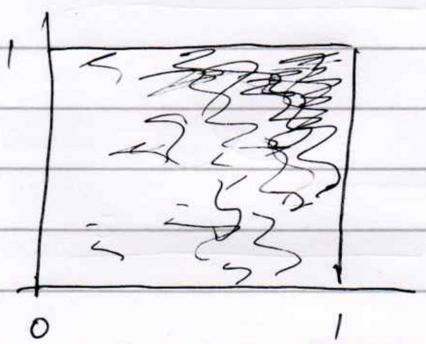
$\Rightarrow P(A) = P(A - A \cap B) + P(A \cup B)$ ②

Use ① + ② to eliminate $P(A - A \cap B)$

$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B)$

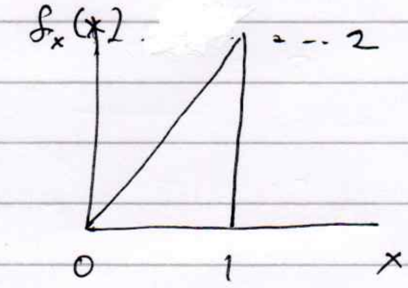
2) Example w/ joint, marginal, conditional pdfs

Consider $f(x,y) = 4xy$ $\quad 0 \leq x \leq 1$, $0 \leq y \leq 1$



- Marginal: $f_x(x) = \int_0^1 4xy \, dy$

$$= 2x, \quad 0 \leq x \leq 1$$



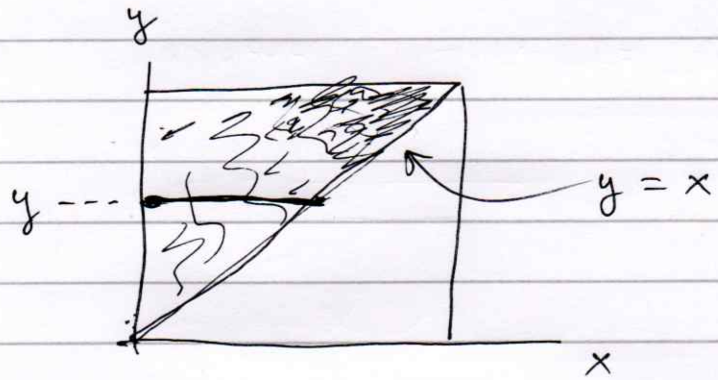
By symmetry, $f_y(y) = 2y, \quad 0 \leq y \leq 1$

- Conditional pdf: $f(x|y) = \dfrac{f(x,y)}{f_y(y)} = \dfrac{4xy}{2y}$

$$= 2x, \quad 0 \leq x \leq 1$$

$\Rightarrow$ X + Y independent ($f(x|y)$ indep. of y)

Example 3) $f(x,y) = \begin{cases} 8xy & 0 \le x \le 1 \\ & x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$



$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y 8xy\,dx$$

$$= 4y^3, \quad 0 \le y \le 1$$

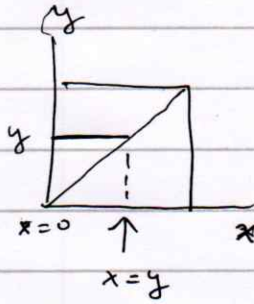
$$f_x(x) = \int_x^1 8xy\,dy = 4xy^2\Big|_x^1$$

$$= 4x(1-x^2), \quad 0 \le x \le 1$$

Example 3, cont.: conditional pdf

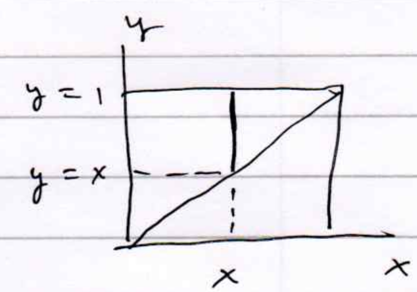
$$f(x|y) = \frac{f(x,y)}{f_y(y)} = \frac{8xy}{4y^3}$$

$$= \frac{2x}{y^2}, \quad 0 \le x \le y$$



$$\ne f(x,y) \Rightarrow x \text{ \& } y \ \underline{\text{not}} \text{ independent}$$

$$f(y|x) = \frac{8xy}{4x(1-x^2)} = \frac{2y}{1-x^2}, \quad x \le y \le 1$$



Check Bayes' theorem:

$$f(x|y) = \frac{f(y|x)\, f_x(x)}{f_y(y)}$$

$$\frac{2x}{y^2} \stackrel{?}{=} \frac{\frac{2y}{\cancel{1-x^2}} \cdot \cancel{4}x\,\cancel{(1-x^2)}}{\cancel{4}y^3}$$

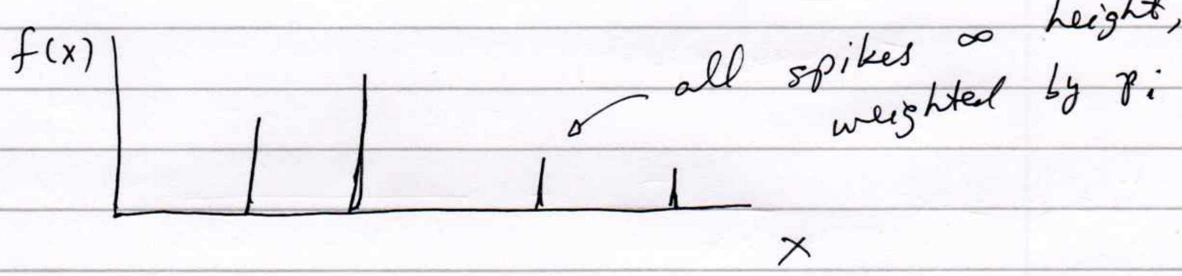
$$= \frac{2x}{y^2} \quad \checkmark$$

δ function $\longleftrightarrow$ pdf / cdf

Suppose discrete r.v. x_i, $i = 1, 2, \ldots$

$$P(x_i) = p_i \qquad \text{(prob. mass func.)}$$

$\Rightarrow$ pdf $\quad f(x) = \sum_i' p_i \, \delta(x - x_i)$



Cumul. dist., use $\displaystyle\int_a^b \delta(x - x') \, dx$

$$= \begin{cases} 1 & a < x' < b \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow F(x) = \int_{-\infty}^{x} f(x') \, dx'$$

$$= \sum_i' p_i \int_{-\infty}^{x} \delta(x' - x_i) \, dx'$$

$$= \sum_{\{i:\, x_i \leq x\}}' p_i$$