Discussion Session – week 2

Example 1

Consider the joint pdf

$$f(x,y) = 1 \,, \quad 0 \le x \le 1 \,, \quad 0 \le y \le 1$$

a) Find pdf of $z = xy$

In lectures we showed

$$g(z) = \int f\left(x, \frac{z}{x}\right) \frac{dx}{x} \qquad \text{(Mellin convolution)}$$

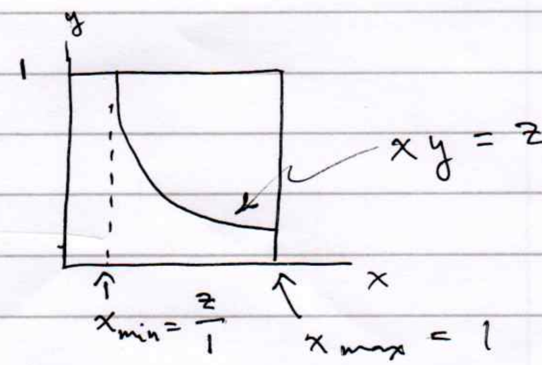
$$= \int_{x_{min}}^{x_{max}} 1 \cdot \frac{dx}{x}$$

$f(x,y)$ is nonzero for $0 \le x \le 1$, $0 \le y \le 1$

$$\Rightarrow \quad 0 \le \frac{z}{x} \le 1 \quad \Rightarrow \quad 0 \le z \le x$$

$$\Rightarrow \quad x_{min} = z$$

$$x_{max} = 1$$



$$\Rightarrow g(z) = \int_z^1 \frac{dx}{x} = \ln x \Big|_z^1 = -\ln z \,, \qquad 0 < z \le 1$$

b) Alternative method – let

$$z = xy \qquad\qquad x = u$$
$$\Rightarrow$$
$$u = x \qquad\qquad y = \frac{z}{u}$$

Jacobian is

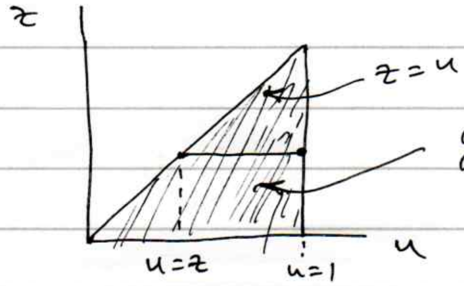
$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{u} & -\frac{z}{u^2} \end{vmatrix} = -\frac{1}{u}$$

$$g(z,u) = |J|\, f(x(z,u), y(z,u))$$

$$= \frac{1}{u}, \quad 0 \le u \le 1, \quad 0 \le z \le u$$

Because: $0 \le x \le 1 \Rightarrow 0 \le u \le 1$

$0 \le y \le 1 \Rightarrow 0 \le \frac{z}{u} \le 1 \Rightarrow 0 \le z \le u$



$g(z,u) = \frac{1}{u}$ in triangle

$$g_z(z) = \int g(z,u)\, du = \int_z^1 \frac{du}{u} = -\ln z \qquad 0 < z \le 1$$

$\uparrow\ y \le 1 \Rightarrow \frac{z}{u} \le 1 \Rightarrow z \le u$

Example 2 – error propagation

Consider r.v.s

$$x_i : \mu_i = 10, \quad \sigma_i = 1, \quad i = 1, 2$$

$$\& \ \mathrm{cov}[x_i, x_j] = 0$$

and let $y = \dfrac{x_1^2}{x_2}$

$$V[y] \approx \left(\frac{\partial y}{\partial x_1}\right)^2\Bigg|_{\vec{x}=\vec{\mu}} \sigma_1^2 + \left(\frac{\partial y}{\partial x_2}\right)^2\Bigg|_{\vec{x}=\vec{\mu}} \sigma_2^2$$

$$= \left(\frac{2x_1}{x_2}\right)^2\Bigg|_{\vec{x}=\vec{\mu}} \sigma_1^2 + \left(-\frac{x_1^2}{x_2^2}\right)^2\Bigg|_{\vec{x}=\vec{\mu}} \sigma_2^2$$

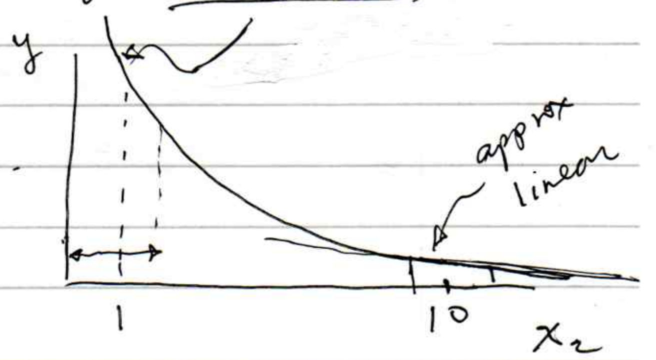
$$= \frac{4\mu_1^2}{\mu_2^2}\sigma_1^2 + \frac{\mu_1^4}{\mu_2^4}\sigma_2^2$$

$$= 4 \times 1 + 1 \times 1 = 5 \Rightarrow \sigma_y = \sqrt{5} \approx 2.2$$

Suppose $\mu_1 = 10, \ \mu_2 = \underline{\underline{1}}$ $\quad (\sigma_1 = \sigma_2 = 1)$

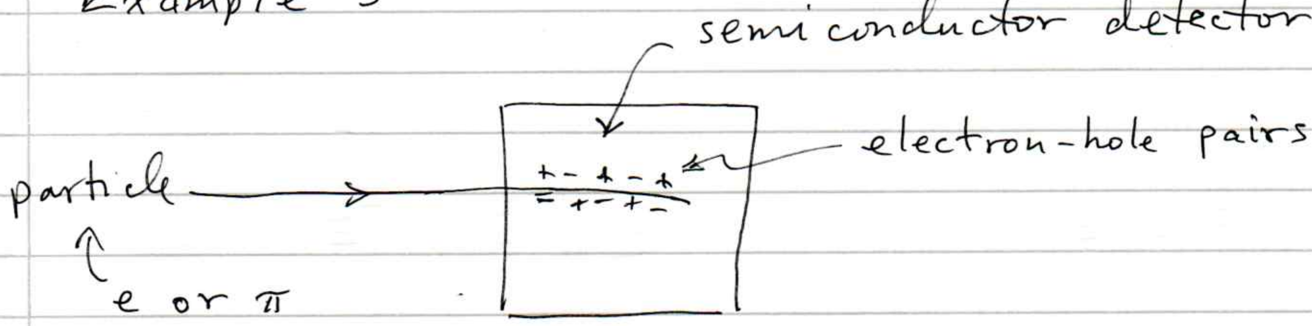
Then $y = \dfrac{x_1^2}{x_2}$ is signficantly <u>non linear</u> in a region of $\sim \pm\sigma_2$

& therefore linear err. prop. is poor approx.



$(\sigma_y \rightarrow \sqrt{10\,400} = 102.0)$

Example 3



of e^-/hole pairs $n \sim \text{Poisson}(\nu)$

If particle $= e$, $\nu = \nu_e$, prior prob $= \pi_e$

" " " π, $\nu = \nu_\pi$, " " $= \pi_\pi$
$= 1 - \pi_e$

From law of total probability,

$$P(n) = P(n|\nu_\pi)\,\pi_\pi + P(n|\nu_e)\,\pi_e$$

$$= \frac{\nu_\pi^n}{n!} e^{-\nu_\pi} \pi_\pi + \frac{\nu_e^n}{n!} e^{-\nu_e} \pi_e$$

The expectation value of n is

$$E[n] = \sum_{n=0}^{\infty} n\,P(n)$$

$$= \pi_\pi \underbrace{\sum_{n=0}^{\infty} n\,P(n|\nu_\pi)}_{= E[n|\nu_\pi] = \nu_\pi} + \pi_e \underbrace{\sum_{n=0}^{\infty} n\,P(n|\nu_e)}_{= E[n|\nu_e] = \nu_e}$$

$$= \pi_\pi \nu_\pi + \pi_e \nu_e$$

To find the variance $V[n] = E[n^2] - (E[n])^2$

first find

$$E[n^2] = \sum_{n=0}^{\infty} n^2 \left(P(n|\nu_\pi)\,\pi_\pi + P(n|\nu_e)\,\pi_e \right)$$

$$= \pi_\pi \, E[n^2|\nu_\pi] + \pi_e \, E[n|\nu_e]$$

Use fact that

$$E[n^2] = V[n] + (E[n])^2$$

and for Poisson var. $V[n] = E[n]$

$$\Rightarrow \quad E[n^2|\nu_i] = \nu_i + \nu_i^2 \quad , \qquad i = \pi, e$$

Assembling the ingredients,

$$V[n] = \pi_\pi (\nu_\pi + \nu_\pi^2) + \pi_e (\nu_e + \nu_e^2)$$
$$- (\pi_\pi \nu_\pi + \pi_e \nu_e)^2$$

Example 4 – proof that covariance matrix $V_{ij} = \mathrm{cov}[x_i, x_j]$ is positive semi-definite,

i.e. $\vec{z}^T V \vec{z} \geq 0$ for any $\vec{z} \in \mathbb{R}^n$

Can transform r.v.s to have mean $\rightarrow 0$ (i.e. let $x_i \rightarrow x_i - \mu_i$) so that

$$V = E[\vec{x}\,\vec{x}^T] \quad , \quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Let $\vec{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n$ (const.)

$$\vec{z}^T V \vec{z} = \vec{z}^T E[\vec{x}\,\vec{x}^T]\,\vec{z}$$

$$= E[\vec{z}^T \vec{x}\,\vec{x}^T \vec{z}] \qquad \text{since } E[\;] \text{ linear}$$

$$= E[(\vec{x}^T \vec{z})^T (\vec{x}^T \vec{z})] \qquad \text{since } A^T B = ((A^T B)^T)^T = (B^T A)^T$$

$$= \underbrace{E[(\vec{z}^T \vec{x})^2]}_{\text{real scalar}} \geq 0$$

Q.E.D.

For e.g. $z_i = \delta_{ij} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}$ — position j

$$\Rightarrow E[x_j^2] = V[x_j] \geq 0.$$