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Problem Sheet 7 Sol'ns 2021

1) $$f(x;\theta) = \frac{x^2}{2\theta^3} e^{-x/\theta} , \quad x \geq 0, \quad \theta > 0$$

$$E[x] = 3\theta , \quad V[x] = 3\theta^2$$

Given i.i.d. sample $x_1, \ldots, x_n$

For (a) - (c) assume n constant.

1a) $$L(\theta) = \prod_{i=1}^{n} \frac{x_i^2}{2\theta^3} e^{-x_i/\theta}$$

$$\ln L(\theta) = \sum_{i=1}^{n} \left[\ln x_i^2 - \ln 2 - \ln \theta^3 - \frac{x_i}{\theta} \right]$$

$$= -3n \ln \theta - \frac{1}{\theta} \sum_{i=1}^{n} x_i + C$$

$$\frac{\partial \ln L}{\partial \theta} = -\frac{3n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} x_i \overset{\text{set}}{=} 0$$

$$\Rightarrow \quad \hat{\theta} = \frac{1}{3n} \sum_{i=1}^{n} x_i$$

b) $$E[\hat{\theta}] = E\left[\frac{1}{3n}\sum_{i=1}^{n} x_i\right]$$

$$= \frac{1}{3n}\sum_{i=1}^{n} \underbrace{E[x_i]}_{3\theta} = \theta$$

$$\Rightarrow \quad b = E[\hat{\theta}] - \theta = 0$$

$$V[\hat{\theta}] = V\left[\frac{1}{3n}\sum_{i=1}^{n} x_i\right]$$

$$= \frac{1}{9n^2}\sum_{i=1}^{n} \underbrace{V[x_i]}_{3\theta^2} = \frac{\theta^2}{3n}$$

$$MVB = -\frac{\left(1+\frac{\partial b}{\partial \theta}\right)^2}{E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right]}$$

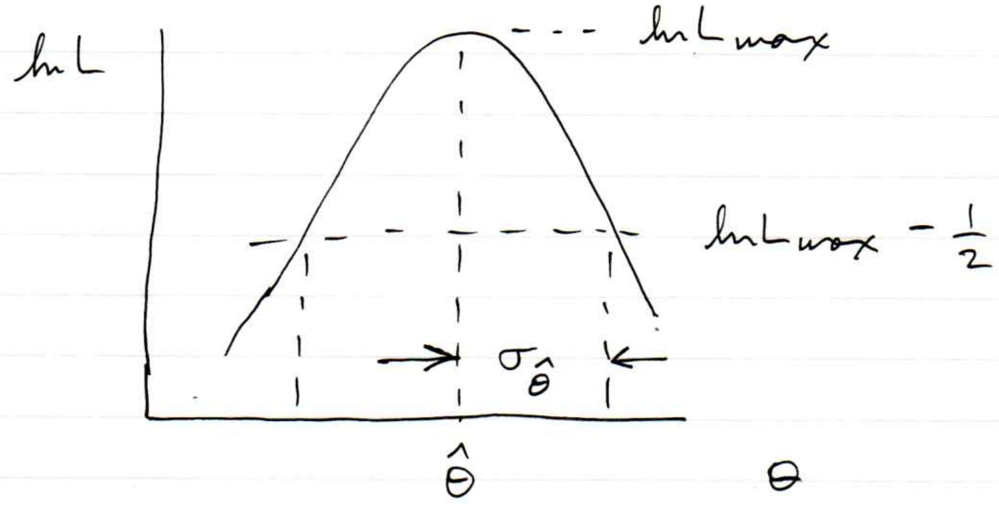
$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{3n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i$$

$$E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = \frac{3n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} \underbrace{E[x_i]}_{3\theta} = -\frac{3n}{\theta^2}$$

$$\Rightarrow \quad MVB = -\frac{(1+0)^2}{\left(-\frac{3n}{\theta^2}\right)} = \frac{\theta^2}{3n}$$

= same as $V[\hat{\theta}]$ $\Rightarrow$ $\hat{\theta}$ is efficient.

1c)



For (d)–(f), treat $n \sim \text{Poisson}(\nu)$ i.e. $P(n|\nu) = \frac{\nu^n e^{-\nu}}{n!}$

$$\nu = \alpha \theta^3 \qquad (\alpha \text{ known})$$

1 (d)

$$L(\theta) = \frac{(\alpha\theta^3)^n}{n!} e^{-\alpha\theta^3} \prod_{i=1}^{n} \frac{x_i}{2\theta^3} e^{-x_i/\theta}$$

$$\ln L(\theta) = \cancel{3n\ln\theta} - \alpha\theta^3 - \cancel{3n\ln\theta} - \sum_{i=1}^{n} \frac{x_i}{\theta} + C$$

$$= -\alpha\theta^3 - \sum_{i=1}^{n} \frac{x_i}{\theta} + C$$

$$\frac{\partial \ln L}{\partial \theta} = -3\alpha\theta^2 + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i \overset{\text{set}}{=} 0$$

$$\Rightarrow \quad \hat{\theta} = \left(\frac{1}{3\alpha} \sum_{i=1}^{n} x_i \right)^{1/4}$$

1e) $$E[a(n,\vec{x})] = \sum_{n=0}^{\infty} \int a(n,\vec{x})\, P(n,\vec{x})\, d\vec{x}$$

Joint probability for n & $\vec{x}$ is

$$P(n,\vec{x}) = f(\vec{x}\,|\,n)\, P(n)$$

$$\Rightarrow E[a(n,\vec{x})] = \sum_{n=0}^{\infty} P(n) \underbrace{\int a(n,\vec{x})\, f(\vec{x}\,|\,n)\, d\vec{x}}$$

$$= \sum_{n=0}^{\infty} P(n)\, E_{\vec{x}}[a(n,\vec{x})\,|\,n]$$

$$= E_n\Big[E_{\vec{x}}[a(n,\vec{x})\,|\,n] \Big]$$

1f) $$\frac{\partial^2 \ln L}{\partial \theta^2} = -6\alpha\theta - \frac{2}{\theta^3} \sum_{i=1}^{n} x_i \quad \Big\}$$ function of both n and $\vec{x}$

$$E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = -6\alpha\theta - \frac{2}{\theta^3} E\left[\sum_{i=1}^{n} x_i\right]$$

use d) $$E\left[\sum_{i=1}^{n} x_i\right] = E_n\left[E_{\vec{x}}\left[\sum_{i=1}^{n} x_i \,\Big|\, n\right]\right] = E_n\Big[\sum_{i=1}^{n} \underbrace{E[x_i]}_{=\,3\theta}\Big]$$

$$= E_n[3n\theta] = 3\underbrace{\nu}_{\alpha\theta^3}\theta = 3\alpha\theta^4$$

$$\Rightarrow E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = -6\alpha - \frac{2}{\theta^3} 3\alpha\theta^4 = -12\alpha\theta$$

$$\Rightarrow V[\hat{\theta}] = \frac{1}{12\alpha\theta} = \frac{\theta^2}{12\nu}$$ compare to $\theta^2/3n$ for fixed n case.

use $\alpha = \nu/\theta^3$

2. The outcome of a measurement consists of two independent random values, x and y, that follow the pdfs

$$
\begin{aligned}
f(x|\theta_1, \theta_2) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\theta_1-\theta_2)^2/2\sigma^2} \ , \\
g(y|\theta_2) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-(y-\theta_2)^2/2\sigma^2} \ .
\end{aligned}
$$

Consider the standard deviation σ (same for x and y) to be known.

(a) Write down the log-likelihood function for θ_1 and θ_2.

[4]

(b) Show that the Maximum-Likelihood estimators for θ_1 and θ_2 are

$$
\begin{aligned}
\hat{\theta}_1 &= x - y \ , \\
\hat{\theta}_2 &= y \ .
\end{aligned}
$$

[6]

(c) Show that the estimators given above are unbiased. Find their exact variances, their covariance and correlation coefficient.

[10]

(d) Show with the help of a sketch how the standard deviations of $\hat{\theta}_1$ and $\hat{\theta}_2$ can be determined from a contour of the log-likelihood function. Label all of the relevant features.

[6]

(e) Suppose θ_1 is the parameter of interest and we regard θ_2 as a nuisance parameter. Find the profiled value $\hat{\hat{\theta}}_2(\theta_1)$ and using this show that the log of the profile likelihood for θ_1 can be written

$$\ln L_{\rm p}(\theta_1) = -\frac{1}{4}\frac{(x-y-\theta_1)^2}{\sigma^2} + C$$

where C represents terms that do not depend on the unknown parameters.

[8]

Show that the variance of $\hat{\theta}_1$ as determined directly from the second derivative of the profile likelihood is the same as found in (c).

[6]

x, y indep. 2014 Exam #2

$$\begin{cases} f(x \mid \theta_1, \theta_2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\theta_1-\theta_2)^2/2\sigma^2} \\ g(y \mid \theta_2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(y-\theta_2)^2/2\sigma^2} \end{cases}$$

a) $L(\theta_1, \theta_2) = P(x, y \mid \theta_1, \theta_2) = f(x \mid \theta_1, \theta_2)\, g(y \mid \theta_2)$

$$= \frac{1}{2\pi\sigma^2} \exp\left[- \frac{(x-\theta_1-\theta_2)^2 + (y-\theta_2)^2}{2\sigma^2} \right]$$

$$\ln L(\theta_1, \theta_2) = -\frac{1}{2}\left[\frac{(x-\theta_1-\theta_2)^2 + (y-\theta_2)^2}{\sigma^2} \right]$$

b) $$\frac{\partial \ln L}{\partial \theta_1} = \frac{x - \theta_1 - \theta_2}{\sigma^2} \overset{\text{set}}{=} 0$$

$$\frac{\partial \ln L}{\partial \theta_2} = \underbrace{\frac{x - \theta_1 - \theta_2}{\sigma^2}}_{\text{cancel}} + \frac{y - \theta_2}{\sigma^2} \overset{\text{set}}{=} 0$$

$$\Rightarrow \boxed{\begin{aligned} \theta_2 &= y \\ \theta_1 &= x - y \end{aligned}}$$

c) $E[\hat{\theta}_1] = E[x] - E[y] = \theta_1 + \theta_2 - \theta_2 = \theta_1$ ✓

$E[\hat{\theta}_2] = E[y] = \theta_2$

$V[\hat{\theta}_1] = V[x] + V[y] = 2\sigma^2$

$V[\hat{\theta}_2] = V[y] = \sigma^2$

c (cont.)

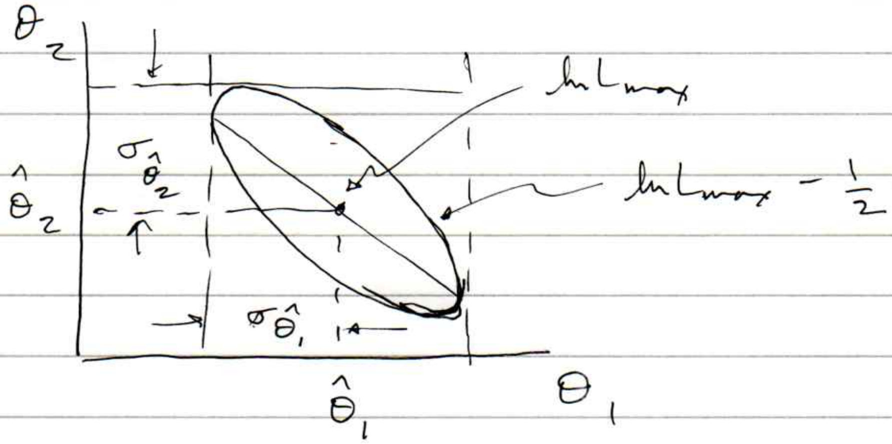
$$\text{cov}[\hat{\theta}_1, \hat{\theta}_2] = \text{cov}[x-y, y]$$

$$= \underbrace{\cancel{\text{cov}[x,y]}}_{0} - \text{cov}[y,y] \qquad \text{(show)}$$

$$= -\sigma^2$$

$$\Rightarrow \rho_{12} = -\frac{\sigma^2}{\sqrt{2}\,\sigma\cdot\sigma} = -\frac{1}{\sqrt{2}}$$

d)



e) $\hat{\hat{\theta}}_2(\theta_1)$ from sol'n to $\frac{\partial \ln L}{\partial \theta_2} = 0$

$$\frac{x - \theta_1 - \theta_2}{\cancel{\sigma^2}} + \frac{y - \theta_2}{\cancel{\sigma^2}} = 0$$

$$\hat{\hat{\theta}}_2(\theta_1) = \frac{1}{2}(x + y - \theta_1)$$

$$\Rightarrow \ln L_p(\theta_1) = -\frac{1}{2}\left[\frac{(x - \theta_1 - \frac{1}{2}(x+y-\theta_1))^2 + (y - \frac{1}{2}(x+y-\theta_1))^2}{\sigma^2}\right]$$

$$\ln L_p(\theta_1) = -\frac{1}{2\sigma^2}\left[\left(\frac{x}{2} - \frac{y}{2} - \frac{\theta_1}{2}\right)^2 + \left(\frac{y}{2} - \frac{x}{2} + \frac{\theta_1}{2}\right)^2\right]$$

$$= -\frac{1}{4}\frac{(x - y - \theta_1)^2}{\sigma^2}$$

$$\frac{\partial^2 \ln L_p}{\partial \theta_1^2} = -\frac{1}{2\sigma^2}$$

$$\Rightarrow \quad -E\left[\frac{\partial^2 \ln L}{\partial \theta_1^2}\right] = \frac{1}{2\sigma^2}$$

$$\Rightarrow \quad V[\hat{\theta}_1] \approx I^{-1}(\theta) = 2\sigma^2$$ (← same as before.)