Discussion Session Notes – Week 3

Example 1 : "Memorylessness" of exponential

Exponential pdf $f(x;\xi) = \frac{1}{\xi} e^{-x/\xi}$, $x \geq 0$

First find cumulative distribution

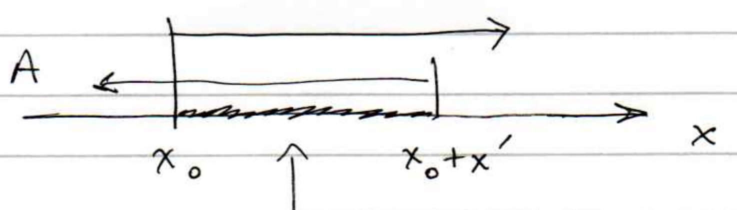
$$F(x) = \int_0^x \frac{1}{\xi} e^{-x'/\xi} \, dx' = -e^{-x'/\xi} \Big|_0^x = 1 - e^{-x/\xi}$$

Next, find $P(x < x_0 + x' \mid x > x_0)$

will show this is $P(x < x')$

Recall $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

For $P(\underbrace{x < x_0 + x'}_{A} \mid \underbrace{x > x_0}_{B})$



$A \cap B = x_0 < x < x_0 + x'$

$$\Rightarrow \quad P(x < x_0 + x' \mid x > x_0) = \frac{P(x_0 < x < x_0 + x')}{P(x > x_0)}$$

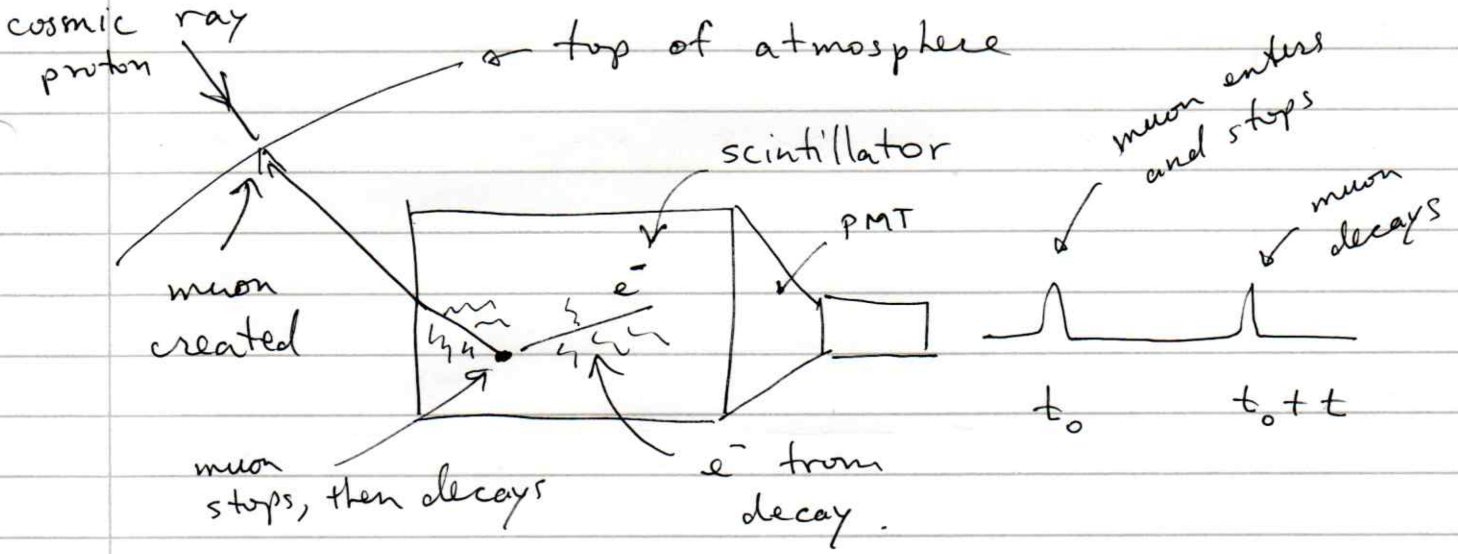
$$= \frac{\int_{x_0}^{x_0+x'} \frac{1}{\xi} e^{-x/\xi}\,dx}{\int_{x_0}^{\infty} \frac{1}{\xi} e^{-x/\xi}\,dx} = \frac{F(x_0 + x') - F(x_0)}{1 - F(x_0)}$$

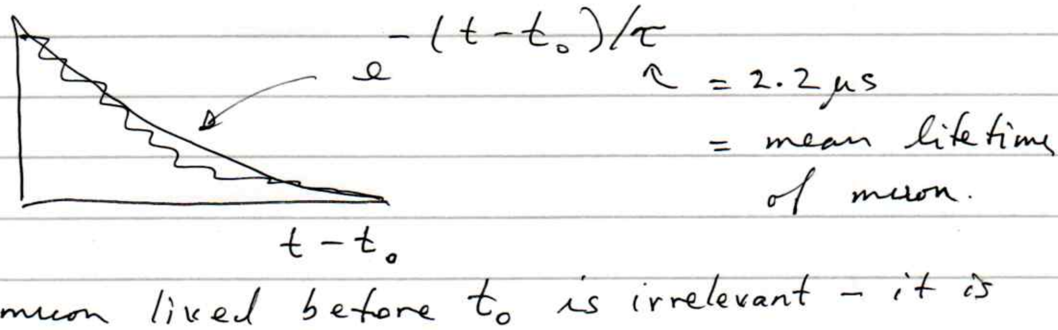
$$F(x) = 1 - e^{-x/\xi}$$

$$= \frac{-e^{-(x_0+x')/\xi} + e^{-x_0/\xi}}{e^{-x_0/\xi}}$$

$$= 1 - e^{-x'/\xi} = F(x') = P(x \le x')$$

And from this $\Rightarrow \quad f(x - x_0 \mid x > x_0) = f(x)$





Time that muon lived before t_0 is irrelevant – it is as "young" at t_0 as when is was born: $f(t - t_0 \mid t > t_0) = f(t)$

<u>Example 2</u> Log-normal dist. + variable trans.

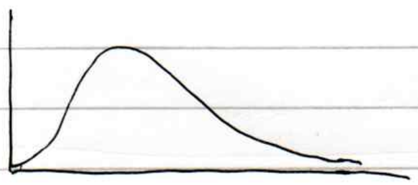
Gaussian: $f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$

Let $y = e^x$ ← find pdf of y

$x = \ln y \qquad \frac{dx}{dy} = \frac{1}{y}$

$$g(y) = f(x(y))\left|\frac{dx}{dy}\right| = \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma y} \exp\left[-\frac{(\ln y-\mu)^2}{2\sigma^2}\right]}_{\text{"log-normal pdf}}$$

(i.e. $\ln y \sim$ Gauss)



$y \quad (y > 0)$

μ, σ^2 are mean, variance of Gaussian x, not of the log-normal y. Can find

$E[y] = \exp\left[\mu + \frac{\sigma^2}{2}\right], \quad V[y] = \left[e^{\sigma^2} - 1\right]\exp(2\mu+\sigma^2)$

$x = \sum_{i=1}^{\text{many}} u_i \xrightarrow{\text{CLT}} x \sim \text{Gauss}$

$y = e^x = \exp\left[\sum_i u_i\right] = \prod_i e^{u_i} \xrightarrow{\text{CLT}} \text{log-normal}$

<u>Sum</u> of many terms $\xrightarrow{\text{CLT}}$ Gauss

Product " " factors $\xrightarrow{\text{CLT}}$ log-normal

Example 3 MC transformation method

pdf $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$ Cauchy

cumulative dist. $F(x) = \int_{-\infty}^{x} \frac{dx'}{\pi(1+x'^2)}$

$$\Rightarrow F(x) = \frac{1}{\pi} \tan^{-1} x' \Big|_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left(\tan^{-1} x + \frac{\pi}{2}\right)$$

$\overset{\text{set}}{=}$ r and solve for x

$r \sim U[0,1]$

$$x(r) = \tan\left[\pi\left(r - \tfrac{1}{2}\right)\right]$$

i.e. if $r_1, r_2, \ldots$ indep. & $\sim U[0,1]$

then $x_i = x(r_i)$ indep. & $\sim f(x) = \frac{1}{\pi(1+x^2)}$

Code: cauchyMC.py

cauchyMC.ipynb