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SDA - Discussion Session Notes - Week 5

Problem sheet 2:

1) $$f(x,y) = \frac{1}{\pi R^2} \qquad x^2 + y^2 \leq R^2$$

Define $$u = \sqrt{x^2+y^2} \qquad u \geq 0$$

$$v = \tan^{-1}\frac{y}{x} \qquad 0 \leq v \leq 2\pi$$

a) Find pdf $g(u,v)$

Use inverse $$x = u \cos v$$

$$y = u \sin v$$

Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u \cos v \end{vmatrix}$$

$$= u\cos^2 v + u \sin^2 v = u$$

$$\Rightarrow g(u,v) = |J| \, f(x(u,v), y(u,v))$$

$$= \frac{u}{\pi R^2}, \qquad 0 \leq u \leq R$$

$g(u,v)$ is independent of v

$\Rightarrow$ joint pdf factorizes

$\Rightarrow$ u and v are independent.

1 b) Marginal pdfs of u & v:

$$g_u(u) = \int g(u,v)\, dv$$

$$= \int_0^{2\pi} \frac{u}{\pi R^2}\, dv$$

$$= \frac{2u}{R^2}, \qquad 0 \le u \le R$$

$$g_v(v) = \int g(u,v)\, du$$

$$= \int_0^{R} \frac{u}{\pi R^2}\, du$$

$$= \frac{1}{\pi R^2} \left. \frac{u^2}{2} \right|_0^R$$

$$= \frac{1}{2\pi}, \qquad 0 \le v \le 2\pi$$

2) $$E\left[c_0 + \sum_{i=1}^{n'} c_i x_i\right] = \int \left(c_0 + \sum_{i=1}^{n'} c_i x_i\right) f(\vec{x})\, d\vec{x}$$

$$= c_0 \underbrace{\int f(\vec{x})\, d\vec{x}}_{=1} + \sum_{i=1}^{n'} c_i \underbrace{\int x_i f(\vec{x})\, d\vec{x}}_{=E[x_i]}$$

$$= c_0 + \sum_{i=1}^{n'} c_i E[x_i]$$

$$V\left[c_0 + \sum_{i=1}^{n'} c_i x_i\right] = E\left[\left(c_0 + \sum_{i=1}^{n'} c_i x_i\right)^2\right] - \left(E\left[c_0 + \sum_{i=1}^{n'} c_i x_i\right]\right)^2$$

$$= E\left[\left(c_0 + \sum_{i=1}^{n'} c_i x_i\right)\left(c_0 + \sum_{j=1}^{n'} c_j x_j\right)\right]$$

$$- \left(c_0 + \sum_{i=1}^{n'} c_i E[x_i]\right)\left(c_0 + \sum_{j=1}^{n'} c_j x_j\right)$$

$$= \cancel{c_0^2} + \cancel{2c_0 \sum_{i=1}^{n'} c_i E[x_i]} + \sum_{i,j=1}^{n'} c_i c_j E[x_i x_j]$$

$$- \cancel{c_0^2} - \cancel{2c_0 \sum_{i=1}^{n'} c_i E[x_i]} - \sum_{i,j=1}^{n'} c_i c_j E[x_i] E[x_j]$$

$$= \sum_{i,j=1}^{n'} c_i c_j \left(E[x_i x_j] - E[x_i] E[x_j]\right)$$

$$= \sum_{i,j=1}^{n'} c_i c_j \operatorname{cov}[x_i, x_j]$$ (indep. of c_0)

2 cont.)

If the x_i are uncorrelated,

$$\mathrm{cov}[x_i, x_j] = \delta_{ij}\, \sigma_i^2$$

$$\Rightarrow \quad V\left[c_0 + \sum_{i=1}^{n} c_i x_i\right] = \sum_{i,j=1}^{n} c_i c_j \,\mathrm{cov}[x_i, x_j]$$

$$= \sum_{i,j=1}^{n} c_i c_j \delta_{ij}\, \sigma_i^2$$

$$= \sum_{i=1}^{n} c_i^2 \sigma_i^2$$

3) We are given

$$V[\alpha x + y] = \alpha^2 \sigma_x^2 + \sigma_y^2 + 2\alpha \sigma_x \sigma_y \geq 0$$

Let $\alpha = \dfrac{\sigma_y}{\sigma_x}$

$$\Rightarrow \frac{\sigma_y^2}{\sigma_x^2}\sigma_x^2 + \sigma_y^2 + 2\left(\frac{\sigma_y}{\sigma_x}\right)\rho\,\sigma_x\sigma_y \geq 0$$

$$\Rightarrow 2\sigma_y^2 + 2\rho\,\sigma_y^2 \geq 0 \quad \Rightarrow \quad \rho \geq -1$$

Let $\alpha = -\sigma_y / \sigma_x$

$$\left(\frac{-\sigma_y}{\sigma_x}\right)^2 \sigma_x^2 + \sigma_y^2 - 2\,\frac{\sigma_y}{\sigma_x}\rho\,\sigma_x\sigma_y \geq 0$$

$$\Rightarrow \sigma_y^2 + \sigma_y^2 - 2\rho\,\sigma_y^2 \geq 0 \quad \Rightarrow \quad \rho \leq 1$$

$$\Rightarrow \boxed{-1 \leq \rho \leq 1}$$