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SDA / PH4515 Discussion Notes - Week 10

See extra slides (w10) for an exercise

on Bayesian parameter estimation & MCMC

Prob Sheet 7 solutions:

1) $f(x;\theta) = \frac{x}{\theta^2} e^{-x/\theta}$, $x \geq 0$, $\theta > 0$

$E[x] = 2\theta$, $V[x] = 2\theta^2$ } given

i.i.d. sample $x_1, \ldots, x_n$

For (a) - (c) assume n constant

1a) $L(\theta) = \prod_{i=1}^{n} \frac{x_i}{\theta^2} e^{-x_i/\theta}$

$$\ln L(\theta) = \sum_{i=1}^{n} \left[\ln x_i - \ln \theta^2 - \frac{x_i}{\theta} \right]$$

$$= -2n \ln\theta - \frac{1}{\theta} \sum_{i=1}^{n} x_i + C$$

C ← terms indep. of θ

$$\frac{\partial \ln L}{\partial \theta} = -\frac{2n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} x_i \overset{\text{set}}{=} 0$$

$$\Rightarrow \hat{\theta} = \frac{1}{2n} \sum_{i=1}^{n} x_i$$

1b) $$E[\hat{\theta}] = E\left[\frac{1}{2n}\sum_{i=1}^{n} x_i\right]$$

$$= \frac{1}{2n}\sum_{i=1}^{n} \underbrace{E[x_i]}_{\substack{\shortparallel \\ 2\theta}} = \underline{\theta}$$

$$\Rightarrow \quad b = E[\hat{\theta}] - \theta = 0$$

$$V[\hat{\theta}] = V\left[\frac{1}{2n}\sum_{i=1}^{n} x_i\right]$$

$$= \frac{1}{4n^2}\sum_{i=1}^{n} \underbrace{V[x_i]}_{\substack{\shortparallel \\ 2\theta^2}} = \underline{\frac{\theta^2}{2n}}$$

$$MVB = -\frac{\left(1+\frac{\partial b}{\partial \theta}\right)^2}{E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right]}$$

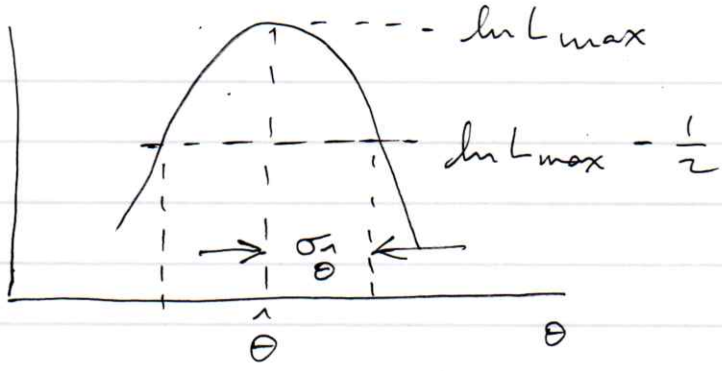
$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{2n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i$$

$$E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = \frac{2n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} \underbrace{E[x_i]}_{2\theta} = -\frac{2n}{\theta^2}$$

$$\frac{\partial b}{\partial \theta} = 0 \quad \text{so}$$

$$MVB = \frac{\theta^2}{2n} \longleftarrow \text{equal to } V[\hat{\theta}]$$

1c)



For (d) − (f) treat $n \sim \text{Poisson}(\nu)$

with $E[n] = \nu = \alpha \theta^2$ ← known const.

$$P(n|\nu) = \frac{\nu^n}{n!} e^{-\nu}$$

1d) $$L(\theta) = \frac{(\alpha\theta^2)^n}{n!} e^{-\alpha\theta^2} \prod_{i=1}^{n} \frac{x_i}{\theta^2} e^{-x_i/\theta}$$

$$\ln L(\theta) = \cancel{2n\ln\theta} + n\ln\alpha - \ln n! - \alpha\theta^2$$

$$\cancel{-2n\ln\theta} + \sum_{i=1}^{n}\left(\ln x_i - \frac{x_i}{\theta}\right)$$

$$= -\alpha\theta^2 - \frac{1}{\theta}\sum_{i=1}^{n} x_i + C$$

$$\frac{\partial \ln L}{\partial \theta} = -2\alpha\theta + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i \overset{\text{set}}{=} 0$$

$$\Rightarrow \quad \hat{\theta} = \left(\frac{1}{2\alpha}\sum_{i=1}^{n} x_i\right)^{1/3}$$

1e) $E[a(n,\vec{x})] = \sum_{n=0}^{\infty} \int a(n,\vec{x})\, P(n,\vec{x})\, d\vec{x}$

Joint prob $P(n,\vec{x}) = f(\vec{x}|n)\, P(n)$

$$\Rightarrow E[a(n,\vec{x})] = \sum_{n=0}^{\infty} P(n) \underbrace{\int a(n,\vec{x})\, f(\vec{x}|n)\, d\vec{x}}$$

$$= \sum_{n=0}^{\infty} P(n)\, E_{\vec{x}}\left[a(n,\vec{x})\,|\,n\right]$$

$$= E_n\left[E_{\vec{x}}\left[a(n,\vec{x})\,|\,n\right]\right] \quad \text{QED}$$

1f) $$\frac{\partial^2 \ln L}{\partial \theta^2} = -2\alpha - \frac{2}{\theta^3}\sum_{i=1}^{n} x_i$$

$$E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = -2\alpha - \frac{2}{\theta^3} E\left[\sum_{i=1}^{n} x_i\right]$$ ← [ ] is func. of n and $\vec{x}$

$$E\left[\sum_{i=1}^{n} x_i\right] = E_n\left[E_{\vec{x}}\left[\sum_{i=1}^{n} x_i \,|\, n\right]\right] = E_n\left[\sum_{i=1}^{n} \underbrace{E[x_i]}_{=\,2\theta}\right]$$

$$= E_n[2n\theta] = 2\underbrace{\nu}_{\alpha\theta^2}\theta = 2\alpha\theta^3$$

$$\Rightarrow E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] = -2\alpha - \frac{2}{\theta^3}\cdot 2\alpha\theta^3 = -6\alpha$$

$$\Rightarrow V[\hat{\theta}] = \frac{1}{6\alpha} = \frac{\theta^2}{6\nu}$$

use $\nu = \alpha\theta^2$

assumes bias ≈ 0 and $V[\hat{\theta}] \approx$ MVB

i.e. here $V[\hat{\theta}]$ smaller than $\theta^2/2n$ (for $n \to \nu$) found with n fixed, since additional info on θ comes from n, since $n \sim$ Poisson$(\alpha\theta^2)$.