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Discussion Session – week 2

Example 1

Consider the joint pdf

$$f(x,y) = 1 \quad , \quad 0 \le x \le 1 \; , \; 0 \le y \le 1$$

a) Find pdf of $z = xy$

In lectures we found

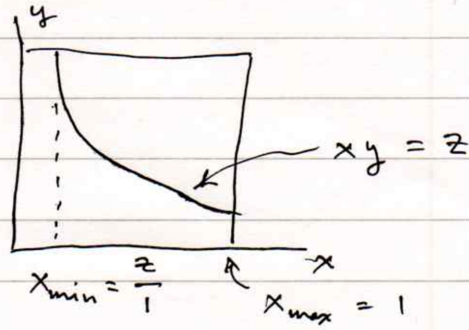
$$g(z) = \int f\left(x, \frac{z}{x}\right) \frac{dx}{x} \qquad \text{(Mellin convolution)}$$

$$= \int_{x_{min}}^{x_{max}} 1 \cdot \frac{dx}{x}$$

$f(x,y)$ is nonzero for $0 \le x \le 1$, $0 \le y \le 1$

$$\Rightarrow 0 \lesssim \frac{z}{x} \lesssim 1 \quad \Rightarrow \quad 0 \lesssim z \lesssim x$$

$$\Rightarrow x_{min} = z$$
$$x_{max} = 1$$



$$\Rightarrow g(z) = \int_z^1 \frac{dx}{x} = \ln x \Big|_z^1$$

$$= -\ln z \quad , \quad 0 < z \lesssim 1$$

b) Alternative method – let

$$z = xy \qquad\qquad x = u$$
$$\Rightarrow$$
$$u = x \qquad\qquad y = \frac{z}{u}$$

Jacobian is

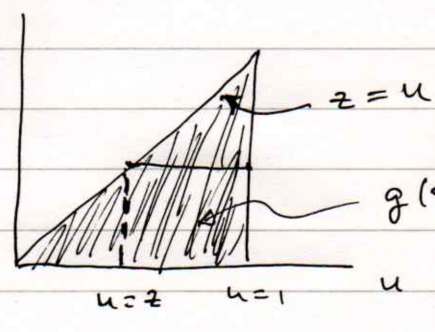
$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \frac{1}{u} & -\frac{z}{u^2} \end{vmatrix} = -\frac{1}{u}$$

$$g(z,u) = |J| \, f(x(z,u), y(z,u))$$

$$= \frac{1}{u}, \quad 0 \le u \le 1, \quad 0 \le z \le u$$

Because: $0 \le x \le 1 \Rightarrow 0 \le u \le 1$

$0 \le y \le 1 \Rightarrow 0 \le \frac{z}{u} \le 1 \Rightarrow 0 \le z \le u$



$$g_z(z) = \int g(z,u)\, du = \int_z^1 \frac{du}{u} = -\ln z, \quad 0 < z \le 1$$

$\uparrow$ $u \ge z$ (see above)

Example 2 – error propagation

Consider r.v.s x_i, $i = 1, 2$

with $\mu_i = 10$, $\sigma_i = 1$, $\mathrm{cov}[x_i, x_j] = 0$

and let $y = \dfrac{x_1^2}{x_2}$. Find variance $V[y]$.

$$V[y] \approx \left(\frac{\partial y}{\partial x_1}\right)^2\Bigg|_{\vec{x}=\vec{\mu}} \sigma_1^2 + \left(\frac{\partial y}{\partial x_2}\right)^2\Bigg|_{\vec{x}=\vec{\mu}} \sigma_2^2$$

$$= \left(\frac{2x_1}{x_2}\right)^2\Bigg|_{\vec{x}=\vec{\mu}} \sigma_1^2 + \left(-\frac{x_1^2}{x_2^2}\right)^2\Bigg|_{\vec{x}=\vec{\mu}} \sigma_2^2$$

$$= \frac{4\mu_1^2}{\mu_2^2}\sigma_1^2 + \frac{\mu_1^4}{\mu_2^4}\sigma_2^2$$

$$= 4 \times 1 + 1 \cdot 1 = \underline{5} \quad \Rightarrow \quad \sigma_y = \sqrt{5} \approx 2.2$$

Now suppose $\mu_1 = 10$, $\mu_2 = 1$ $(\sigma_1 = \sigma_2 = 1)$

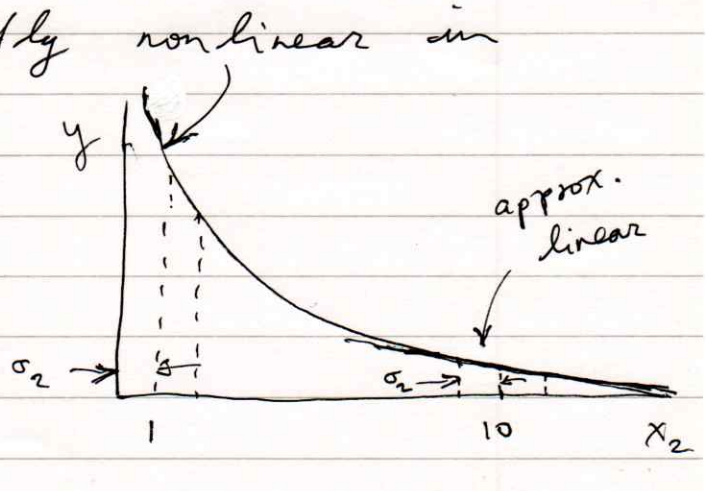
Then $y = \dfrac{x_1^2}{x_2}$ is significantly nonlinear in

a region of $\sim \pm\sigma_2$

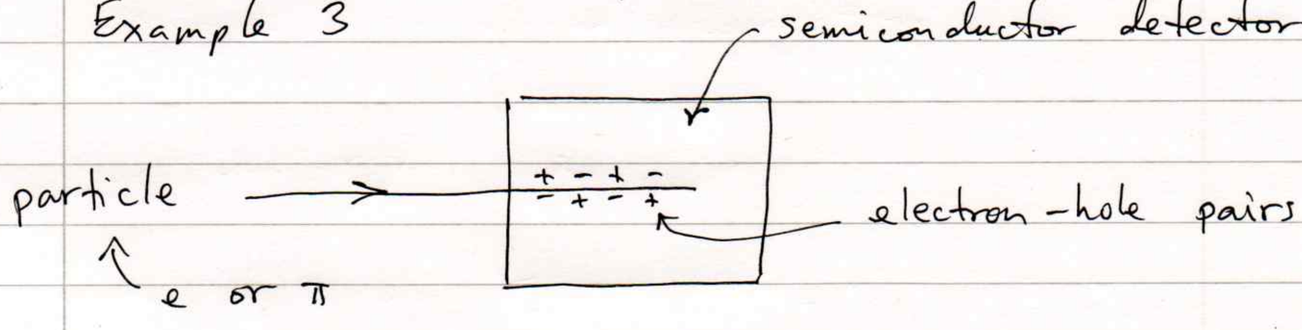
& therefore linear err. prop.

is poor approx.

$(\sigma_y \to \sqrt{10400} = 102.0)$



Example 3



of e^-/hole pairs n $\sim \text{Poisson}(\nu_e)$ for e

$\sim \text{Poisson}(\nu_\pi)$ for π

Incident particles are mixture of e, π

w/ relative fractions π_e, $\underbrace{\pi_\pi}_{= 1 - \pi_e}$ (prior probs.)

From law of total probability

$$P(n) = P(n | \nu_\pi)\, \pi_\pi + P(n | \nu_e)\, \pi_e$$

$$= \frac{\nu_\pi^n}{n!} e^{-\nu_\pi} \pi_\pi + \frac{\nu_e^n}{n!} e^{-\nu_e} \pi_e$$

Expectation value of n is

$$E[n] = \sum_{n=0}^{\infty} n\, P(n)$$

$$= \pi_\pi \underbrace{\sum_{n=0}^{\infty} n\, P(n | \nu_\pi)}_{= E[n | \nu_\pi] = \nu_\pi} + \pi_e \underbrace{\sum_{n=0}^{\infty} n\, P(n | \nu_e)}_{= E[n | \nu_e] = \nu_e}$$

$$= \pi_\pi \nu_\pi + \pi_e \nu_e$$

Find variance $V[n] = E[n^2] - (E[n])^2$

First find

$$E[n^2] = \sum_{n=0}^{\infty} n^2 \left(P(n|\nu_\pi)\pi_\pi + P(n|\nu_e)\pi_e \right)$$

$$= \pi_\pi E[n^2|\nu_\pi] + \pi_e E[n^2|\nu_e]$$

Use fact that

$$E[n^2] = V[n] + (E[n])^2$$

and for Poisson var. $V[n] = E[n]$

$$\Rightarrow E[n^2|\nu_i] = \nu_i + \nu_i^2 , \qquad i = \pi, e$$

Assembling the ingredients,

$$V[n] = \pi_\pi(\nu_\pi + \nu_\pi^2) + \pi_e(\nu_e + \nu_e^2)$$

$$- (\pi_\pi \nu_\pi + \pi_e \nu_e)^2$$

Example 4 – proof that covariance matrix

$V_{ij} = \mathrm{cov}[x_i, x_j]$ is positive semi-definite

i.e. $\vec{z}^T V \vec{z} \geq 0$ for any $\vec{z} \in \mathbb{R}^n$

Can transform r.v.s to have mean of zero

i.e. let $x_i \rightarrow x_i - \mu_i$ so that

$$V = E[\vec{x}\,\vec{x}^T] \quad , \quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

"outer product"

Let $\vec{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{R}^n$ (const.)

$$\vec{z}^T V \vec{z} = \vec{z}^T E[\vec{x}\,\vec{x}^T]\,\vec{z}$$

$$= E[\vec{z}^T \vec{x}\,\vec{x}^T \vec{z}]$$ since $E[\ ]$ linear

$$= E[(\vec{x}^T z)^T (\vec{x}^T \vec{z})]$$ since $A^T B = ((A^T B)^T)^T = (B^T A)^T$

$$= E[(\vec{z}^T \vec{x})^2] \geq 0$$

real scalar

Q.E.D.

For e.g. $z_i = \delta_{ij} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ on position j

$$\Rightarrow E[x_j^2] = V[x_j] \geq 0 .$$