G. Cowan 17.10.22

Discussion Session – week 3 1

Example 1: "memorylessness" of exponential

Exponential pdf $f(x;\xi) = \frac{1}{\xi} e^{-x/\xi}$, $x \geq 0$

First, find cumulative distribution

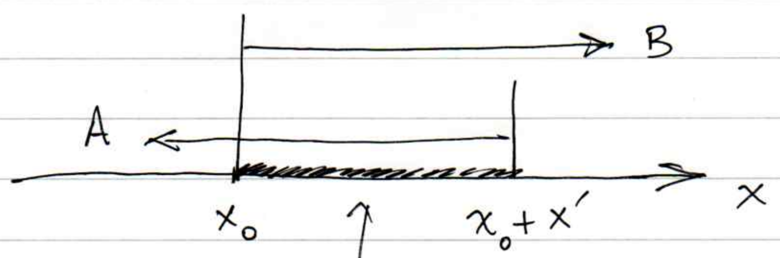
$$F(x) = \int_0^x \frac{1}{\xi} e^{-x'/\xi} dx' = -e^{-x'/\xi}\Big|_0^x = 1 - e^{-x/\xi}$$

Next, find $P(x < x_0 + x' \mid x > x_0)$

↖ will show this is $P(x < x')$

Recall $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

For $P(x < x_0 + x' \mid x > x_0)$



$A \cap B = x_0 < x < x_0 + x'$

$$\Rightarrow P(x < x_0 + x' \mid x > x_0) = \frac{P(x_0 < x < x_0 + x')}{P(x > x_0)}$$

$$= \frac{\int_{x_0}^{x_0+x'} \frac{1}{\xi} e^{-x/\xi} dx}{\int_{x_0}^{\infty} \frac{1}{\xi} e^{-x/\xi} dx} = \frac{F(x_0 + x') - F(x_0)}{1 - F(x_0)}$$

$$F(x_0) = 1 - e^{-x_0/\xi}$$

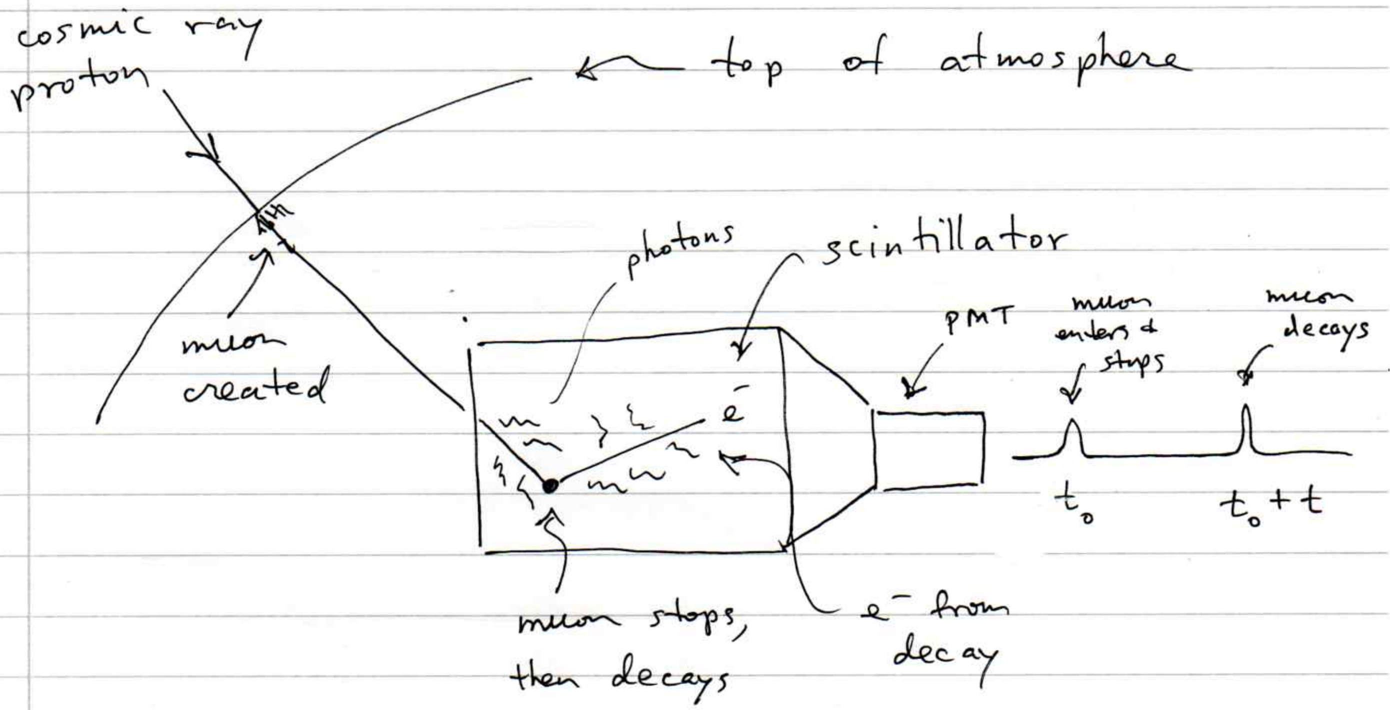
$$= \frac{-e^{-(x_0+x')/\xi} + e^{-x_0/\xi}}{e^{-x_0/\xi}}$$

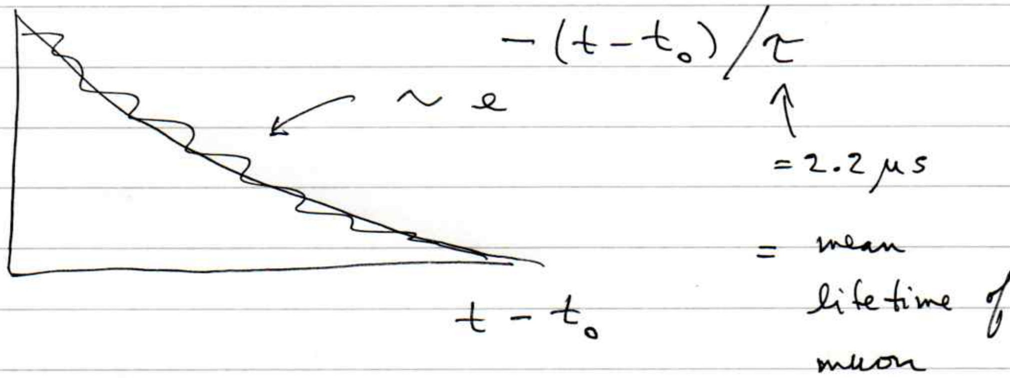
$$= 1 - e^{-x'/\xi} = F(x') = P(x \leq x')$$

And from this using $f(x) = \frac{\partial F}{\partial x}$

$$f(x - x_0 \mid x > x_0) = f(x)$$

Example "memoriless" exponential





Time that muon lived before t_0 is irrelevant. Muon is just as "young" at t_0 as when it was first born:

$$f(t - t_0 \mid t > t_0) = f(t)$$

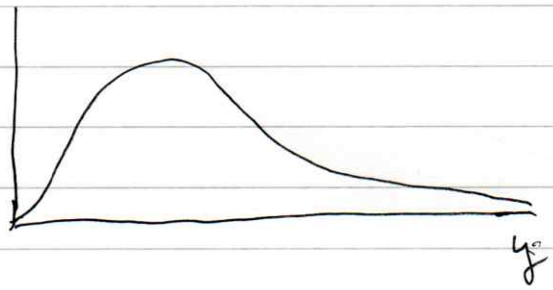
Example 3 Log-normal dist. & variable trans.

Gaussian $f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$

Let $y = e^x$ & find pdf of y

$$x = \ln y \,, \quad \frac{dx}{dy} = \frac{1}{y}$$

$$g(y) = f(x(y)) \left| \frac{dx}{dy} \right| = \frac{1}{\sqrt{2\pi}\,\sigma\, y} \exp\left[-\frac{(\ln y - \mu)^2}{2\sigma^2} \right]$$



$(y > 0)$

μ, σ^2 are mean, variance of Gaussian x, not of the log-normal y. Can find

$$E[y] = \exp\left[\mu + \frac{\sigma^2}{2}\right], \quad V[y] = \left[e^{\sigma^2} - 1\right] \exp(2\mu + \sigma^2)$$

$$x = \sum_{i=1}^{\text{many}} u_i \xrightarrow{\text{CLT}} x \sim \text{Gauss}$$

$$y = e^x = \exp\left[\sum_i u_i\right] = \prod e^{u_i} \xrightarrow{\text{CLT}} \text{log-normal}$$

Sum of many terms $\xrightarrow{\text{CLT}}$ Gauss

Product " " factors $\xrightarrow{\text{CLT}}$ log-normal

Example 4 MC transformation method

Cauchy pdf $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$

Cumulative dist. $F(x) = \int_{-\infty}^{x} \frac{dx'}{\pi(1+x'^2)}$

$\Rightarrow F(x) = \frac{1}{\pi} \tan^{-1} x' \Big|_{-\infty}^{x}$

$= \frac{1}{\pi}\left(\tan^{-1} x + \frac{\pi}{2}\right)$

$\overset{\text{set}}{=} r$ and solve ~~for~~ x

$r \sim U[0,1]$

$\Rightarrow x(r) = \tan\left[\pi\left(r - \frac{1}{2}\right)\right]$

i.e if $r_1, r_2, \ldots$ indep. & $\sim U[0,1]$

then $x_i = x(r_i)$ indep & $\sim \frac{1}{\pi(1+x^2)}$

Code: cauchyMC.py

cauchyMC.ipynb