GDC 31.10.22
SDA / PH4515

Discussion Notes – week 5

Problem sheet 2

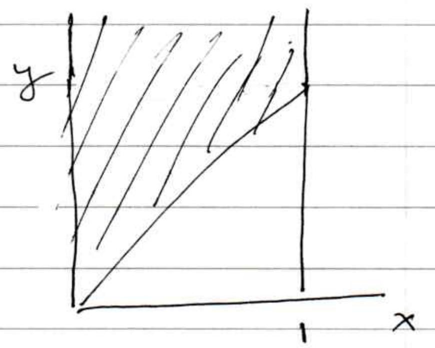
1) $f(x,y) = 6x(1-x)\frac{1}{\xi} e^{-(y-x)/\xi}$

for $0 \le x \le 1$, $x \le y < \infty$ and $\xi > 0$

1a) $f(x,y)$ does not factorize as

$f_x(x) f_y(y) \Rightarrow x, y$ not independent.

(Because restriction on y depends on x:



)

1b) Let $u = y - x$, $v = x$ $\Rightarrow$ $x = v$, $y = u + v$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$\Rightarrow$ joint pdf of u, v

$$g(u,v) = |J| \, f(x(u,v), y(u,v))$$

$$= 6v(1-v)\frac{1}{\xi} e^{-u/\xi}$$

$0 \le x \le 1 \Rightarrow 0 \le v \le 1$

$x \le y < \infty$

$\hookrightarrow v \le u + v < \infty \Rightarrow u \ge 0$

factorizes $\Rightarrow u, v$ are indep.

c) $g_u(u) = \int g(u,v)\,dv$

$$= \frac{1}{\xi} e^{-u/\xi} \underbrace{\int_0^1 6v(1-v)\,dv}_{\left(\frac{6v^2}{2} - \frac{6v^3}{3}\right)\Big|_0^1 = 1}$$

$$= \frac{1}{\xi} e^{-u/\xi}$$

$g_v(v) = \int g(u,v)\,du$

$$= \underbrace{\int_0^\infty \frac{1}{\xi} e^{-u/\xi}\,du}_{=1} \times 6v(1-v)$$

$$= 6v(1-v), \quad 0 \le v \le 1$$

(So indeed $g(u,v) = g_u(u)\, g_v(v)$.)

2) $$E\left[c_0 + \sum_{i=1}^{n'} c_i x_i\right] = \int \left(c_0 + \sum_{i=1}^{n'} c_i x_i\right) f(\vec{x})\, d\vec{x}$$

$$= c_0 \underbrace{\int f(\vec{x})\, d\vec{x}}_{=1} + \sum_{i=1}^{n'} c_i \underbrace{\int x_i\, f(\vec{x})\, d\vec{x}}_{= E[x_i]}$$

$$= c_0 + \sum_{i=1}^{n'} c_i\, E[x_i]$$

$$V\left[c_0 + \sum_{i=1}^{n'} c_i x_i\right] = E\left[\left(c_0 + \sum_{i=1}^{n'} c_i x_i\right)^2\right] - \left(E\left[c_0 + \sum_{i=1}^{n'} c_i x_i\right]\right)^2$$

$$= E\left[\left(c_0 + \sum_{i=1}^{n'} c_i x_i\right)\left(c_0 + \sum_{j=1}^{n'} c_j x_j\right)\right]$$

$$- \left(c_0 + \sum_{i=1}^{n'} c_i E[x_i]\right)\left(c_0 + \sum_{j=1}^{n'} c_j E[x_j]\right)$$

$$= c_0^2 + 2c_0 \sum_{i=1}^{n'} c_i E[x_i] + \sum_{i,j=1}^{n'} c_i c_j\, E[x_i x_j]$$

$$- c_0^2 - 2c_0 \sum_{i=1}^{n'} c_i E[x_i] - \sum_{i,j=1}^{n'} c_i c_j\, E[x_i]\, E[x_j]$$

$$= \sum_{i,j=1}^{n'} c_i c_j \left(E[x_i x_j] - E[x_i]\, E[x_j]\right)$$

$$= \sum_{i,j=1}^{n'} c_i c_j \operatorname{cov}[x_i, x_j] \quad (\text{indep. of } c_0)$$

2 (cont.)

If the x_i are uncorrelated,

$$\mathrm{cov}[x_i, x_j] = \delta_{ij}\, \sigma_i^2$$

$$V\left[c_0 + \sum_{i=1}^{n} c_i x_i\right] = \sum_{i,j=1}^{n} c_i c_j \,\mathrm{cov}[x_i, x_j]$$

$$= \sum_{i,j=1}^{n} c_i c_j \delta_{ij} \sigma_i^2$$

$$= \sum_{i=1}^{n} c_i^2 \sigma_i^2$$

3) We are given

$$V[\alpha x + y] = \alpha^2 \sigma_x^2 + \sigma_y^2 + 2\alpha \sigma_x \sigma_y \rho \geq 0$$

Let $\alpha = \dfrac{\sigma_y}{\sigma_x}$

$$\Rightarrow \frac{\sigma_y^2}{\sigma_x^2} \sigma_x^2 + \sigma_y^2 + 2\left(\frac{\sigma_y}{\sigma_x}\right) \rho \sigma_x \sigma_y \geq 0$$

$$\Rightarrow 2\sigma_y^2 + 2\rho\sigma_y^2 \geq 0 \quad \Rightarrow \quad \underline{\rho \geq -1}$$

Let $\alpha = -\sigma_y / \sigma_x$

$$\left(-\frac{\sigma_y}{\sigma_x}\right)^2 \sigma_x^2 + \sigma_y^2 - 2\frac{\sigma_y}{\sigma_x} \rho \sigma_x \sigma_y \geq 0$$

$$\Rightarrow \sigma_y^2 + \sigma_y^2 - 2\rho\sigma_y^2 \geq 0 \quad \Rightarrow \quad \rho \leq 1$$

$$\Rightarrow \boxed{-1 \leq \rho \leq 1}$$