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Discussion Notes Week 6

Prob Sheet 3 sol'ns.

1) $x \sim U[\alpha, \beta]$

i.e. $f(x) = \begin{cases} \dfrac{1}{\beta - \alpha}, & \alpha \le x \le \beta \\ 0, & \text{otherwise} \end{cases}$

$$E\left[\frac{1}{x}\right] = \int \frac{1}{x} f(x)\, dx = \int_\alpha^\beta \frac{1}{x} \frac{dx}{\beta - \alpha}$$

$$= \frac{1}{\beta - \alpha} \ln \frac{\beta}{\alpha} \qquad \alpha = 1, \; \beta = 2$$

$$= \frac{1}{2-1} \ln \frac{2}{1} = 0.693$$

$$E[x] = \int_\alpha^\beta \frac{x}{\beta - \alpha}\, dx = \frac{1}{\beta - \alpha} \left. \frac{x^2}{2} \right|_\alpha^\beta$$

$$= \frac{1}{2} \frac{(\beta^2 - \alpha^2)}{\beta - \alpha} = \frac{1}{2}(\alpha + \beta)$$

$$= \frac{1}{2}(1+2) = \frac{3}{2}$$

$$\Rightarrow \frac{1}{E[x]} = \frac{2}{3} \quad \text{not eq. to } E\left[\frac{1}{x}\right]$$

2) $$P(\vec{n} \mid \vec{p}, N) = \frac{N!}{n_1! \cdots n_M!} p_1^{n_1} \cdots p_M^{n_M}$$

multinomial

$$E[\vec{n}] = N \vec{p}$$

$$\mathrm{cov}[n_i, n_j] = N p_i (\delta_{ij} - p_j)$$

Let $u \equiv \sum_{k=1}^{K} n_k$, $K < M$

Using error propagation,

$$V[u] = \sum_{i,j=1}^{M} \left. \frac{\partial u}{\partial n_i} \frac{\partial u}{\partial n_j} \right|_{\vec{n} = E[\vec{n}]} \mathrm{cov}[n_i, n_j]$$

$$\frac{\partial u}{\partial n_i} = \begin{cases} 1, & i \leq K \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow V[u] = \sum_{i,j=1}^{K} 1 \cdot N p_i (\delta_{ij} - p_j)$$

$$= N \left[\sum_{i=1}^{K} p_i - \left(\sum_{i=1}^{K} p_i \right) \left(\sum_{j=1}^{K} p_j \right) \right]$$

$$= N p_K (1 - p_K)$$

= variance for binomial

where $p_K \equiv \sum_{i=1}^{K} p_i$

3) x follows $f(x)$

cdf is $F(x) \Rightarrow f(x) = \frac{dF}{dx}$

$r \sim U[0,1]$

i.e. $g(r) = 1 \quad , \quad 0 \leq r \leq 1$

pdf

If $F(x) = r \Rightarrow x = F^{-1}(r)$

pdf of $x(r)$ is

$$p(x) = g(r) \left| \frac{dr}{dx} \right| = \frac{g(r)}{\left| \frac{dx}{dr} \right|}$$

$$\frac{dx}{dr} = \underbrace{\frac{d}{dr} F^{-1}(r) = \frac{1}{\frac{dF}{dx}(x(r))}}_{} = \frac{1}{f(x(r))}$$

inverse function theorem

$$\Rightarrow \frac{dr}{dx} = f(x)$$

$$\Rightarrow p(x) = \underset{g(r)}{1} \times \underset{\left|\frac{dr}{dx}\right|}{f(x)} = \underline{f(x)}$$

Statistical Data Analysis
Example problem for week 6 discussion
G. Cowan / 7 November 2022

1(a) Consider the test statistic

$$t(\vec{x}) = \ln \frac{f(\vec{x}|\vec{\mu}_1)}{f(\vec{x}|\vec{\mu}_0)} .$$

Show that this $t(\vec{x})$ can be written in the form

$$t(\vec{x}) = w_0 + \sum_{i=1}^{n} w_i x_i ,$$

or equivalently $t(\vec{x}) = w_0 + \vec{w}^T \vec{x}$, where $\vec{w}$ is a column vector of coefficients w_i, $i = 1, \ldots, n$.

1(b) Show that the coefficients in the vector $\vec{w}$ found in (a) satisfy the property of a Fisher discriminant, i.e.,

$$\vec{w} \propto W^{-1}(\vec{\mu}_1 - \vec{\mu}_0) ,$$

where W is the sum of the covariance matrices for the two hypotheses.

1(c) The prior probabilities for hypotheses of $\vec{\mu}_0$ and $\vec{\mu}_1$ are π_0 and π_1, respectively. Show that the posterior probability for $\vec{\mu}_0$ is

$$P(\vec{\mu}_0|\vec{x}) = \frac{1}{1 + \frac{\pi_1}{\pi_0} e^t} ,$$

where $t = \ln[f(\vec{x}|\vec{\mu}_1)/f(\vec{x}|\vec{\mu}_0)]$ as before.

Fisher discriminant w/ Gaussian data

$$\vec{x} \sim \text{Gauss}(\vec{\mu}_k, \underbrace{V}_{\text{cov. matrix}}) , \qquad k = 0, 1$$

$\hookrightarrow (x_1, \ldots, x_n)$

i.e. $$f(\vec{x}|\vec{\mu}_k) = \frac{1}{(2\pi)^{n/2} |V|^{1/2}} \exp\left[-\frac{1}{2}(\vec{x}-\vec{\mu}_k)^T V^{-1} (\vec{x}-\vec{\mu}_k)\right]$$

Let test statistic $t(\vec{x}) = \ln \dfrac{f(\vec{x}|\vec{\mu}_1)}{f(\vec{x}|\vec{\mu}_0)}$

a) $$\ln \frac{f(\vec{x}|\vec{\mu}_1)}{f(\vec{x}|\vec{\mu}_0)} = -\frac{1}{2}\left[(\vec{x}-\vec{\mu}_1)^T V^{-1}(\vec{x}-\vec{\mu}_1) - (\vec{x}-\vec{\mu}_0) V^{-1} (\vec{x}-\vec{\mu}_0)\right]$$

$$= -\frac{1}{2}\Big[\vec{x}^T V^{-1}\vec{x} - \vec{\mu}_1^T V^{-1}\vec{x} - \vec{x}^T V^{-1}\vec{\mu}_1 + \vec{\mu}_1^T V^{-1}\vec{\mu}_1$$

$$- \vec{x}^T V^{-1}\vec{x} + \vec{\mu}_0 V^{-1}\vec{x} + \underbrace{\vec{x}^T V^{-1}\vec{\mu}_0}_{= \vec{\mu}_0^T V^{-1}\vec{x}} - \vec{\mu}_0^T V^{-1}\vec{\mu}_0\Big]$$

since scalar $(\,)^T = (\,)$ and $V^{-1} = (V^{-1})^T$

$$= \underbrace{-\frac{1}{2}\left[\vec{\mu}^T V^{-1}\vec{\mu}_1 - \vec{\mu}_0 V^{-1}\vec{\mu}_0\right]}_{w_0} + \underbrace{(\vec{\mu}_1 - \vec{\mu}_0)^T V^{-1}}_{\vec{w}^T} \vec{x}$$

b)

$$= w_0 + \vec{w}_1^T \vec{x}$$

If $W = V + V \Rightarrow W^{-1} = \frac{1}{2} V^{-1}$

$$\Rightarrow \vec{w} = 2 W^{-1}(\vec{\mu}_1 - \vec{\mu}_0)$$

c) Find $P(H_0 | \vec{x})$

$$= \frac{f(\vec{x} | H_0)\,\pi_0}{f(\vec{x} | H_0)\,\pi_0 + f(\vec{x} | H_1)\,\pi_1}$$

$$= \frac{\pi_0}{\pi_0 + \pi_1 e^{t}}$$

$$t = \ln \frac{f(\vec{x} | H_1)}{f(\vec{x} | H_0)}$$