

Problem sheet 6 solutions:

$$1) \quad n \sim \text{Poisson}(s+b) \quad b = 3.9$$

$$n_{\text{obs}} = 16$$

$$a) \quad p_0 = P(n \geq n_{\text{obs}} \mid s=0, b=3.9)$$

$$= \sum_{n=n_{\text{obs}}}^{\infty} \frac{b^n}{n!} e^{-b}$$

$$= 1 - \sum_{n=0}^{n_{\text{obs}}-1} \frac{b^n}{n!} e^{-b}$$

upper limit of sum

↓

$$= F_{\chi^2}(2b; 2n_{\text{obs}}) \quad \leftarrow \begin{array}{l} n_{\text{dot}} = 2(m+1) \\ = 2n_{\text{obs}} \end{array}$$

$$= \text{scipy.stats.chi2}(2 \times 3.9, 2 \times 16)$$

$$= 1 - \text{TMATH}::\text{Prob}(2 \times 3.9, 2 \times 16)$$

$$= \underline{3.58 \times 10^{-6}}$$

$$b) \quad z_0 = \Phi^{-1}(1 - p_0)$$

$$= \text{scipy.stats.norm.ppf}(1 - p_0)$$

$$= \text{TMATH}::\text{Norm Quantile}(1 - p_0)$$

$$= \underline{4.5}$$

$$2) \quad x \sim \text{Gauss}(\mu, \sigma^2)$$

i.i.d. sample x_1, \dots, x_N

$$2a) \quad L(\mu, \sigma^2) = \prod_{i=1}^N \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}}$$

$$\Rightarrow \ln L(\mu, \sigma^2) = -\frac{1}{2} \sum_{i=1}^N \frac{(x_i - \mu)^2}{\sigma^2} - \frac{N}{2} \ln \sigma^2 + C$$

$$2b) \quad \frac{\partial \ln L}{\partial \mu} = \sum_{i=1}^N \frac{x_i - \mu}{\sigma^2} \stackrel{\text{set}}{=} 0 \quad (1)$$

$$\frac{\partial \ln L}{\partial \sigma^2} = \frac{1}{2} \sum_{i=1}^N \frac{(x_i - \mu)^2}{(\sigma^2)^2} - \frac{N}{2\sigma^2} \stackrel{\text{set}}{=} 0 \quad (2)$$

$$\text{From } (1) \quad \hat{\mu} = \frac{1}{N} \sum_{i=1}^N x_i$$

$$\text{From } (2) \quad \hat{\sigma}^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \hat{\mu})^2$$

2c) Fisher information matrix

$$I_{ij} = -E \left[\frac{\partial^2 \ln L}{\partial \theta_i \partial \theta_j} \right]$$

$$\theta_1 = \mu,$$

$$\theta_2 = \sigma^2$$

$$\frac{\partial^2 \ln L}{\partial \mu^2} = -\frac{N}{\sigma^2}$$

$$\frac{\partial^2 \ln L}{\partial (\sigma^2)^2} = \frac{N}{2\sigma^4} - \sum_{i=1}^N \frac{(x_i - \mu)^2}{\sigma^6}$$

$$\frac{\partial^2 \ln L}{\partial \mu \partial \sigma^2} = -\frac{1}{\sigma^4} \sum_{i=1}^N (x_i - \mu)$$

$$E \left[\frac{\partial^2 \ln L}{\partial \mu^2} \right] = -\frac{N}{\sigma^2}$$

$$E \left[\frac{\partial^2 \ln L}{\partial (\sigma^2)^2} \right] = \frac{N}{2\sigma^4} - \sum_{i=1}^N \frac{E \left[(x_i - \mu)^2 \right]}{\sigma^6} = -\frac{N}{2\sigma^4}$$

$$E \left[\frac{\partial^2 \ln L}{\partial \mu \partial \sigma^2} \right] = -\frac{1}{\sigma^4} \sum_{i=1}^N \underbrace{E \left[x_i - \mu \right]}_0 = 0$$

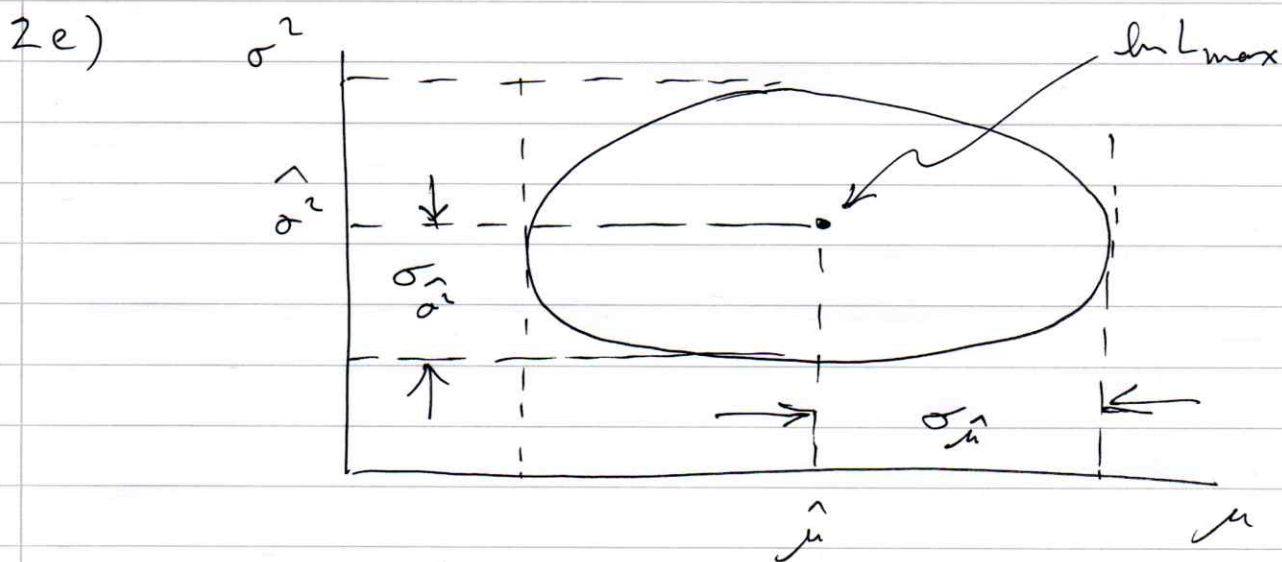
$$I = \begin{pmatrix} \frac{N}{\sigma^2} & 0 \\ 0 & \frac{N}{2\sigma^4} \end{pmatrix}$$

2d) Use $V^{-1} = I$

Assumes zero bias, into inequality \rightarrow equality

Since I diagonal, inverse by inspection

$$V = \begin{pmatrix} \frac{\sigma^2}{N} & 0 \\ 0 & \frac{2\sigma^4}{N} \end{pmatrix} = \begin{pmatrix} V[\hat{\mu}] & \text{cov}[\hat{\mu}, \hat{\sigma}^2] \\ \text{cov}[\hat{\mu}, \hat{\sigma}^2] & V[\hat{\sigma}^2] \end{pmatrix}$$



No tilt, corresponds to $\text{cov}[\hat{\mu}, \hat{\sigma}^2] = 0$