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RHUL Physics
15 December, 2011

Statistics Problems for the NIKHEF Onderzoekschool Subatomaire Fysica (part 3)

Exercise 6: Consider a measured value x that follows a Gaussian distribution with a mean μ and standard deviation σ. Suppose we know σ exactly and that $\mu \geq 0$, and the goal is to set an upper limit on μ given an observed value of x.

(a) Make a sketch of the distributions of x assuming $\mu = 0$ and also assuming some nonzero value of μ. Suppose one has an observed value of x that lies somewhere between 0 and μ. Indicate on the sketch the p-values of μ and of $\mu = 0$ (p_0 and p_μ).

(b) Write down the p-values for μ and $\mu = 0$, p_μ and p_0, as functions of x in terms of the standard normal cumulative distribution Φ.

(c) Suppose we want to construct an upper limit for μ at a confidence level $1 - \alpha$. Recall this is done by equating the p-value for μ equal to α and solving for μ. By doing this with the p_μ found in (b), show that the upper limit for μ is

$$\mu_{\rm up} = x + \sigma \Phi^{-1}(1 - \alpha) \ , \tag{1}$$

where as usual Φ^{-1} is the standard normal quantile.

(d) In the $\mathrm{CL_s}$ procedure, a value of μ is excluded if

$$p'_\mu = \frac{p_\mu}{1 - p_0} \tag{2}$$

is found less than a given value α. By using the p-values from (b), show that this leads to the upper limit

$$\mu_{\rm up} = x + \sigma \Phi^{-1} \left(1 - \alpha \Phi \left(\frac{x}{\sigma} \right) \right) \ . \tag{3}$$

(e) Consider the Bayesian approach to this problem and take a flat prior pdf μ, i.e.,

$$\pi(\mu) = \begin{cases} 1 & \mu \geq 0 \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{4}$$

Find the posterior pdf $p(\mu|x)$, and by requiring

$$1 - \alpha = \int_{-\infty}^{\mu_{\rm up}} p(\mu|x) \, d\mu \tag{5}$$

show that for this problem one obtains the same upper limit as what was found using the $\mathrm{CL_s}$ method, namely,

$$\mu_{\rm up} = x + \sigma \Phi^{-1} \left(1 - \alpha \Phi \left(\frac{x}{\sigma} \right) \right) \ . \tag{6}$$