DRAFT 0.0

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Invariance under parameter transformation with the Jeffreys prior

The Jeffreys prior probability density for a set of parameters $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$ is given by

$$\pi(\boldsymbol{\theta}) \propto \sqrt{\det I(\boldsymbol{\theta})} \tag{1}$$

where the matrix I is the Fisher information, defined by

$$I_{ij}(\boldsymbol{\theta}) = -E\left[\frac{\partial^2 \ln L}{\partial\theta_i \partial\theta_j}\right] \,, \tag{2}$$

and L is the likelihood that specifies the probability for data x given the parameters $\boldsymbol{\theta}$. Note that here we are using the notation L for the likelihood of $\boldsymbol{\theta}$ but we take $L(x|\boldsymbol{\theta})$ also to refer to the probability for the data given $\boldsymbol{\theta}$.

We consider here only the one-parameter case and demonstrate that, under conditions often satisfied in practice, inference based on the Jeffreys prior for the parameters θ is the same as if one transforms to an alternative parameter $\eta(\theta)$.

As a preliminary step, we need to show the relation

$$E\left[\left(\frac{\partial \ln L}{\partial\theta}\right)^2\right] = -E\left[\frac{\partial^2 \ln L}{\partial\theta^2}\right] \,. \tag{3}$$

To do this, we rewrite the left-hand side of (3) as

$$E\left[\left(\frac{\partial \ln L}{\partial\theta}\right)^2\right] = \int L \frac{1}{L}\frac{\partial L}{\partial\theta}\frac{\partial \ln L}{\partial\theta}\, dx \,, \tag{4}$$

Now use the fact that

$$\frac{\partial}{\partial\theta}\left(L\frac{\partial \ln L}{\partial\theta}\right) = L\frac{\partial^2 \ln L}{\partial\theta^2} + \frac{\partial \ln L}{\partial\theta}\frac{\partial L}{\partial\theta} \tag{5}$$

to write

$$
\begin{aligned}
E\left[\left(\frac{\partial \ln L}{\partial \theta}\right)^2\right] &= \int \left[\frac{\partial}{\partial \theta}\left(L\frac{\partial \ln L}{\partial \theta}\right) - L\frac{\partial^2 \ln L}{\partial \theta^2}\right] dx \\
&= \frac{\partial}{\partial \theta}\int L\frac{\partial \ln L}{\partial \theta}\, dx - E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] \\
&= \frac{\partial}{\partial \theta}\int L\,\frac{1}{L}\frac{\partial L}{\partial \theta}\, dx - E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] \\
&= = \frac{\partial^2}{\partial \theta^2}\int L\, dx - E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] \\
&= -E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right] \,, \qquad (6)
\end{aligned}
$$

where the final equality follows from the fact that $\int L\, dx = 1$, since the integral is over the entire data space, and thus its (second) derivative is zero. The relation (3) holds as long as the derivative with respect to θ can be pulled outside of the integral, which means that the range of allowed data values cannot depend on θ.

We can now show that the prior pdf based on the Jeffreys' prior is invariant under a transformation of parameter. Suppose we start with a parameter θ and we base our prior pdf on the Jeffreys prior,

$$
\pi(\theta) \propto \sqrt{-E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right]} \,. \qquad (7)
$$

The posterior pdf is therefore given by Bayes' theorem to be

$$
p(\theta|x) \propto L(x|\theta)\pi(\theta) = L(x|\theta)\sqrt{-E\left[\frac{\partial^2 \ln L}{\partial \theta^2}\right]} = L(x|\theta)\sqrt{E\left[\left(\frac{\partial \ln L}{\partial \theta}\right)^2\right]} \,, \qquad (8)
$$

where the final equality followed from use of Eq. (3).

Now suppose we transform to a new parameter $\eta(\theta)$ with inverse $\theta(\eta)$. Using the usual rules of transformation of pdfs we find

$$
\begin{aligned}
p(\eta|x) &= p(\theta(\eta)|x)\left|\frac{d\theta}{d\eta}\right| \\
&\propto L(x|\theta(\eta))\sqrt{E\left[\left(\frac{\partial \ln L}{\partial \theta}\right)^2\right]} \\
&\propto L(x|\theta(\eta))\sqrt{E\left[\left(\frac{\partial \ln L}{\partial \theta}\frac{\partial \theta}{\partial \eta}\right)^2\right]} \qquad (9)
\end{aligned}
$$

Alternatively we could have used the parameter η from the start. Using the Jeffreys' prior based on η in Bayes' theorem gives the posterior pdf

$$\begin{aligned} p(\eta|x) &\propto L(x|\eta)\sqrt{-E\left[\frac{\partial^2 \ln L}{\partial \eta^2}\right]} \\ &\propto L(x|\eta)\sqrt{E\left[\left(\frac{\partial \ln L}{\partial \eta}\right)^2\right]} \\ &\propto L(x|\eta)\sqrt{E\left[\left(\frac{\partial \ln L}{\partial \theta}\frac{\partial \theta}{\partial \eta}\right)^2\right]} \end{aligned} \tag{10}$$

This leads to the same result as Eq. (9), which shows that inference based on the Jeffreys' prior is invariant under choice of parametrization of the problem.