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Significance from a p-value

This note clarifies two common situations in which one defines a significance Z from a p-value. The two cases involve the significance of a test of a null (e.g., background-only) hypothesis against an alternative where:

1. a data fluctuation in only one direction (e.g., positive observed signal strength) gives increased incompatibility with the null;
2. a data fluctuation in either direction gives increased incompatibility with the null,

where "increased incompatibility withe the null" means increased compatibility with the alternative. Cases 1 and 2 correspond to a one- or two-sided test, respectively.

1. One-sided test

In the first case, one is testing the null hypothesis that no signal is present, and under this assumption the measured signal strength could fluctuate up or down relative to some background level. If the relevant alternative to the null is a positive signal, then the p-value would be defined with this in mind, i.e., data outcomes that are less compatible with the null are those that appear as signal excess. Once this p-value is found, a significance Z is defined such that a standard Gaussian value of Z has an upper-tail area of p, as illustrated in Fig. 1(a). This corresponds to the relation

$$p = 1 - \Phi(Z) \; , \tag{1}$$

where Φ is the standard Gaussian cumulative distribution, or equivalently

$$Z = \Phi^{-1}(1-p) \; , \tag{2}$$

where Φ^{-1} is the standard Gaussian quantile. According to this definition, the significance can take on values from $-\infty < Z < \infty$. If the data come out exactly at the background level, then $p = 0.5$, since one has a 50% probability of an upward (signal-like) fluctuation, and in this case one has $Z = 0$, i.e., there is no evidence against the background-only hypothesis.

The pdf of the p-value is $f(p) = 1, \; 0 \leq p \leq 1$, and so the pdf of Z is

$$g(Z) = f(p) \left| \frac{dp}{dZ} \right| = \varphi(Z) \; , \tag{3}$$

where $\varphi(Z) = d\Phi/dZ$ is the standard Gaussian pdf. A negative value of Z ($p > 0.5$) corresponds to the case where the data fluctuates lower than the expected background level.

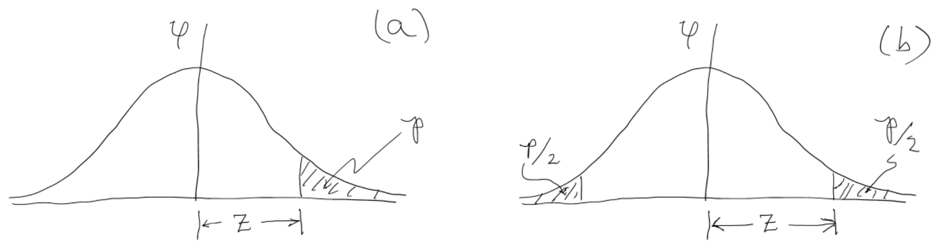

Figure 1: The relation between significance Z and p-value as defined for (a) a one-sided and (b) a two-sided test (see text).

2. Two-sided test

The second case pertains to a situation where any fluctuation, positive or negative, constitutes evidence against the null hypothesis. The p-value in this case would take data outcomes with fluctuations in either direction from the background-expectation as increasingly incompatible with the predictions of the null. One then defines Z according to Fig. 1(b), i.e.,

$$p = 2\,[1 - \Phi(Z)] \tag{4}$$

or

$$Z = \Phi^{-1}(1 - p/2)\ . \tag{5}$$

In this way one has $Z \geq 0$ always. If the data were to come out exactly equal to the background-only expectation, one would have $p = 1$ and $Z = 0$, and any fluctuation positive or negative gives increasing incompatibility with the null.

In case 2, the pdf of Z is found to be

$$g(Z) = 2\varphi(Z)\ , \quad Z \geq 0\ . \tag{6}$$

That is, Z follows a standard Gaussian pdf but only for $Z \geq 0$.