DRAFT 0.0

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Including a limit in a likelihood

Suppose a measurement yields data $y \sim P(y|\mu, \theta)$, where μ is the parameter of interest and θ is a nuisance parameter. If possible one would like to constrain the nuisance parameter with a control measurement u, which in the simplest case could be a Gaussian distributed point estimate of θ with standard deviation σ_u. In such a case the likelihood becomes

$$L(\mu, \theta) = P(y, u|\mu, \theta) = P(y|\mu, \theta) \, \frac{1}{\sqrt{2\pi}\sigma_u} e^{-(u-\theta)^2/2\sigma_u^2} \; . \tag{1}$$

Using a likelihood of this form one can find the maximum-likelihood estimators and confidence regions for all of the parameters.

It may happen, however, that one does not have a point estimate u but rather only an upper limit on θ, $\theta_{\rm up}$, at a specified confidence level $1 - \alpha$. Suppose this is obtained (see, e.g., [1]) from the p-value of θ,

$$p_\theta = P(u \le u_{\rm obs}|\theta) = \int_{-\infty}^{u_{\rm obs}} \frac{1}{\sqrt{2\pi}\sigma_u} e^{-(u-\theta)^2/2\sigma_u^2} \, du \; . \tag{2}$$

The upper limit is the value of θ such that $p_\theta = \alpha$, for which one finds

$$\theta_{\rm up} = u + \sigma_u \Phi^{-1}(1 - \alpha) \; , \tag{3}$$

where Φ^{-1} is the standard normal quantile (inverse of the standard Gaussian cumulative distribution). It is not obvious how to include the information from $\theta_{\rm up}$ into a likelihood as its own sampling distribution depends on θ and σ_u, and we are supposing that σ_u is not reported.

It may often be, however, that an upper limit is reported only if the control measurement is consistent with zero; otherwise u and σ_u would have been reported as a point estimate and standard deviation. We could therefore assume that u came out small, in any case not significantly greater than σ_u. If we suppose $u \approx 0$ then

$$\sigma_u = \frac{\theta_{\rm up}}{\Phi^{-1}(1 - \alpha)} \; . \tag{4}$$

The values $u = 0$ and σ_u from Eq. (4) can then be used in a Gaussian term in the likelihood of Eq. (1) in the same manner as if one had an actual point estimate and standard deviation. Relevant values for the quantiles are, e.g., $\Phi^{-1}(0.90) = 1.28$, $\Phi^{-1}(0.95) = 1.64$.

All of the complications entailed by this procedure are avoided if control measurements are reported with a point estimate and standard deviation, not just an upper limit. The problem is complicated further by the existence of different kinds of limits, flip-flopping between point estimates and limits [2], etc., all of which goes beyond the scope of the present discussion.

References

[1] G. Cowan, K. Cranmer, E. Gross and O. Vitells, EPJC 71 (2011) 1554.

[2] Gary J. Feldman, Robert D. Cousins, *A Unified Approach to the Classical Statistical Analysis of Small Signals*, Phys.Rev.D57:3873-3889,1998.