DRAFT 0.1

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Note on Finding a Measurement from a Limit

Suppose we are given an upper limit $s_{\rm up}$ for the expected number of signal events s at confidence level $1-\alpha$, as well as the expected limit assuming $s = 0$, $E[s_{\rm up}|0]$ and "error band", i.e., the standard deviation $\sigma[s_{\rm up}|0]$. The goal is to find from these values an estimate $\hat{s}$ for s as well as its standard deviation $\sigma_{\hat{s}}$. In contrast to the method of Marumi, the result below only uses the observed and expected limit, not the width of the error band.

The reported limit was based on the p-value

$$p_s = P(\hat{s} \leq \hat{s}_{\rm obs}|s) \ . \tag{1}$$

If $\hat{s}$ is Gaussian distributed with mean s and variance $\sigma^2_{\hat{s}} = V[\hat{s}|s]$, then this is

$$p_s = \Phi\left(\frac{\hat{s}-s}{\sigma_{\hat{s}}}\right) \ , \tag{2}$$

where Φ is the standard Gaussian cumulative distribution. Here $\hat{s}$ refers to the observed value; we drop "obs" to simplify the notation. The upper limit is the value of s such that $p_s = \alpha$, which gives

$$s_{\rm up} = \hat{s} + \Phi^{-1}(1-\alpha)\sigma_{\hat{s}|s_{\rm up}} \ , \tag{3}$$

where $\sigma^2_{\hat{s}|s_{\rm up}} = V[\hat{s}|s_{\rm up}]$ is the variance of $\hat{s}$ under assumption of $s = s_{\rm up}$ and Φ^{-1} is the standard Gaussian quantile.

Solving Eq. (3) for $\hat{s}$ gives

$$\hat{s} = s_{\rm up} + Q\sigma_{\hat{s}|s_{\rm up}} \ , \tag{4}$$

where we define $Q = \Phi^{-1}(1-\alpha)$ (e.g., $Q = 1.64$ for a confidence level $1-\alpha = 0.95$). This does not quite give the desired answer because the standard deviation $\sigma_{\hat{s}|s_{\rm up}}$ is not given. Rather, we have the expected $s_{\rm up}$ as well its the standard deviation of $s_{\rm up}$, under the assumption of $s = 0$. We will denote these below as $E_0 \equiv E[s_{\rm up}|0]$ and $\sigma_0 = \sigma[s_{\rm up}|0]$.

To relate the given ingredients — $s_{\rm up}$, E_0 and σ_0 — to the desired results, suppose the estimate of s results from a counting experiment where a number of events n follows a Poisson distribution with mean $s+b$. Here the expected number of background events b was known when the limit was initially found but it is not an input for the present problem. The maximum-likelihood estimator for s and its variance are

$$\hat{s} = n - b \ , \tag{5}$$

$$V[\hat{s}|s] = s + b \ . \tag{6}$$

Using Eq. (6) for the variance allows us to rewrite Eq. (3) so as to relate $s_{\rm up}$ and $\hat{s}$,

$$(s_{\rm up} - \hat{s})^2 = Q^2 (s_{\rm up} + b) \ . \tag{7}$$

Solving this equation for $s_{\rm up}$

$$s_{\rm up} = \hat{s} + \frac{Q^2}{2} \left[1 + \left(1 + \frac{4(\hat{s} + b)}{Q^2} \right)^{1/2} \right] \ . \tag{8}$$

Or solving instead for $\hat{s}$ gives

$$\hat{s} = s_{\rm up} - Q\sqrt{s_{\rm up} + b} \ . \tag{9}$$

We are given the expected limit assuming $s = 0$, here written E_0. Using Eq. (8) and approximating $E[\sqrt{\hat{s}}] \approx \sqrt{s}$ and then evaluating with $s = 0$ gives

$$E_0 \equiv E[s_{\rm up} | s = 0] \approx \frac{Q^2}{2} \left[1 + \left(1 + \frac{4b}{Q^2} \right)^{1/2} \right] \ . \tag{10}$$

The other given input is σ_0, the standard deviation of $s_{\rm up}$ assuming $s = 0$. By using error propagation with Eq. (8) we can then find the variance of $s_{\rm up}$ in terms of the variance of $\hat{s}$, and then evaluate at $s = 0$. This gives

$$\sigma_0 \equiv \sigma[s_{\rm up} | s = 0] = \sqrt{b} \left[1 + \left(1 + \frac{4b}{Q^2} \right)^{-1/2} \right] \ . \tag{11}$$

Both Eqs. (10) and (11) can be solved in principle for b, and so the system is over-determined. Equation (10) for the expected limit can easily be solved in closed form; the equation for the standard deviation (11) is quartic and thus less easy to handle. If we ignore the information from σ_0 and just solve Eq. (10) for b we obtain

$$b = \frac{E_0^2}{Q^2} - E_0 \ . \tag{12}$$

So by first solving for b and using it in Eq. (9) one can find the estimator $\hat{s}$. Using these together with Eq. (6) gives its standard deviation

$$\sigma_{\hat{s}} = \sqrt{\hat{s} + b} \ . \tag{13}$$