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An apparent paradox from the likelihood ratio

To compare two point hypotheses one can carry out a frequentist statistical test or compute a Bayes factor. For the case of Gaussian data x with two different means μ_0 and μ_1 and the same standard deviation σ, there is an apparent mystery in that the two approaches appear to lead to very different conclusions. Although the frequentist and Bayesian approaches can in general lead to different results, the particular paradox in this case can be resolved, as shown below.

Consider the following two hypotheses:

$$H_0 \quad : \quad x \sim \text{Gauss}(\mu_0, \sigma) \; , \tag{1}$$

$$H_1 \quad : \quad x \sim \text{Gauss}(\mu_1, \sigma) \; . \tag{2}$$

That is, we assume that the measurement can be represented by a single Gaussian distributed variable x with mean μ (equal to either μ_0 or μ_1) and standard deviation σ, so that the likelihood function is

$$L(\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \; . \tag{3}$$

According to the Neyman-Pearson Lemma (see, e.g., Ref. [2]), when defining a test of a given size α of H_0, to obtain the highest power with respect to the alternative H_1 the optimal test statistic is given by the likelihood ratio $L(\mu_1)/L(\mu_0)$. Equivalently we can use a monotonic function of this and therefore we define the statistic

$$q = -2 \ln \frac{L(\mu_1)}{L(\mu_0)} = \frac{(x-\mu_1)^2}{\sigma^2} - \frac{(x-\mu_0)^2}{\sigma^2} = \frac{\mu_1^2 - \mu_0^2 - 2(\mu_1 - \mu_0)x}{\sigma^2} \; . \tag{4}$$

That is, the quadratic terms in (4) cancel and the resulting statistic is a linear function of x. The variable q therefore also follows a Gaussian distribution with expectation value and variance under assumption of a given μ given by

$$E[q|\mu] \quad = \quad \frac{\mu_1^2 - \mu_0^2 - 2(\mu_1 - \mu_0)\mu}{\sigma^2} \; , \tag{5}$$

$$V[q] \quad = \quad \frac{4(\mu_1 - \mu_0)^2}{\sigma^2} \; . \tag{6}$$

These results were given in Eqs. (73) and (74) of Ref. [1] for the special case of $\mu_0 = 0$ and $\mu_1 = \mu$.

The paradox is the following: If one has μ_0 and μ_1 very close to each other, then the two distributions $f(q|H_0)$ and $f(q|H_1)$ are also very close, and therefore the ratio

$$r = \frac{f(q|H_1)}{f(q|H_0)} \tag{7}$$

should be close to one. That is, if one has no sensitivity (equal means) then one should not be able to favour one hypothesis over the other. On the other hand, if one computes this ratio for a certain value of q, say, $q = 4$, then surely the likelihood ratio of the data

$$\frac{L(x|H_1)}{L(x|H_0)} \tag{8}$$

cannot be unity, and in fact from the definition (4) it must be

$$\frac{L(x|H_1)}{L(x|H_0)} = e^{-q/2} \; . \tag{9}$$

So the Bayes factor (here just the likelihood ratio) is $e^{-q/2}$ and may clearly depart from unity and give arbitrarily strong evidence in favour of one hypothesis or the other.

One can show, however, that even in the limit when μ_0 and μ_1 become very close to each other, for a fixed q the likelihood ratio is indeed always given by Eq. (9), and furthermore one obtains exactly the same value for the ratio r from Eq. (7). In the case where the two means μ_0 and μ_1 are exactly equal, then from Eq. (4) one has $q = 0$ and thus the distribution of q is a delta function at zero. This resolves the paradox.

This can be checked explicitly by computing the ratio r from Eq. (7), where the distributions $f(q|H_1)$ and $f(q|H_0)$ are Gaussian with mean and variance given by Eqs. (5) and (6). One finds

$$r = \frac{e^{-(q-E[q|1])^2/2V[q]}}{e^{-(q-E[q|0])^2/2V[q]}} \tag{10}$$

or equivalently

$$-2\ln r = \frac{E[q|1]^2 - E[q|0]^2 - 2(E[q|1] - E[q|0])q}{V[q]} \tag{11}$$

The ingredients are

$$E[q|0] = \frac{\mu_1^2 - \mu_0^2 - 2(\mu_1 - \mu_0)\mu_0}{\sigma^2} \; , \tag{12}$$

$$E[q|1] = \frac{\mu_1^2 - \mu_0^2 - 2(\mu_1 - \mu_0)\mu_1}{\sigma^2} = -E[q|0] \; , \tag{13}$$

$$V[q] = \frac{4(\mu_1 - \mu_0)^2}{\sigma^2} \; , \tag{14}$$

and therefore

$$-2\ln r = \frac{4(\mu_1^2 + \mu_0^2 - 2\mu_0\mu_1)}{\sigma^2} \frac{\sigma^2}{4(\mu_1 - \mu_0)^2} q = q \; . \tag{15}$$

The two likelihood ratios are therefore equal, i.e.,

$$\frac{L(x|H_1)}{L(x|H_0)} = \frac{f(q|H_1)}{f(q|H_0)} = e^{-q/2} \ , \tag{16}$$

and thus at any given q the likelihood ratio is equal to $e^{-q/2}$.

References

[1] Glen Cowan, Kyle Cranmer, Eilam Gross and Ofer Vitells, *Asymptotic formulae for likelihood-based tests of new physics*, Eur. Phys. J. C 71 (2011) 1554.

[2] A. Stuart, J.K. Ord, and S. Arnold, *Kendall's Advanced Theory of Statistics*, Vol. 2A: *Classical Inference and the Linear Model* 6th Ed., Oxford Univ. Press (1999), and earlier editions by Kendall and Stuart.