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Draft 0.5, June 8th, 2011

CLs as a Power Constrained Limit

In the PL frequentist method we test the signal hypothesis, H_μ, (signal with a strength μ), and if the observed p-value with respect to this hypothesis is $< 5\%$, the hypothesis is rejected at the 95% CL. For a given Higgs mass, we find the signal strength μ such that $p_{\mu,obs} = 5\%$ and denote this μ by $\mu_{up}(m_H)$ or $\mu_{95}(m_H)$. If $\mu_{up}(m_H) < 1$, a SM Higgs with mass, m_H is excluded at the 95% CL.

Let $q_{\mu,obs}$ be the observed test statistic. Using the asymptotic formulas of the PL, we find that the p-value is given by

$$p_\mu = 1 - \phi(\sqrt{q_{\mu,obs}}) \tag{1}$$

from which we find ("CL_{s+b}") using $\sqrt{q_{\mu,obs}} = \frac{\mu-x}{\sigma}$

$$\mu_{up} = x + \sigma\phi^{-1}(1-\alpha) \tag{2}$$

where α is the size of the test, chosen to be 5% to set a 95% CL limit. This means that if the p-value is smaller then α the H_μ hypothesis is rejected. To set up an upper limit the p-avlue is set to be equal to α.

In the CLs method, one finds the ratio of p-values

$$CL_s \equiv \frac{CL_{sb}}{CL_b} = \frac{p_\mu}{1-p_b} \tag{3}$$

and finds the signal strength μ_{up,CL_s} such that this ratio is 5% with respect to the observed data. Using the asymptotic equations we find

$$1 - p_b = \phi(\sqrt{q_{\mu,A}} - \sqrt{q_{\mu,obs}}) \tag{4}$$

Inserting $x = 0$ for BG-only Asimov data we find

$$1 - p_b = \phi(\frac{x}{\sigma}) \tag{5}$$

We therefore find that

$$CL_s = \frac{1-\phi(\sqrt{q_{\mu,obs}})}{\phi(\sqrt{q_{\mu,A}} - \sqrt{q_{\mu,obs}})} \tag{6}$$

By requiring $CL_s = \alpha'$ we find that the upper limit with the CLs method equals to

$$\mu_{up}^{CL_s} = x + \sigma\phi^{-1}(1 - \alpha'\phi(\sqrt{q_{\mu,A}} - \sqrt{q_{\mu,obs}})) \tag{7}$$

which yields

$$\mu_{up}^{CL_s} = x + \sigma\phi^{-1}(1 - \alpha'\phi(\frac{x}{\sigma})) \tag{8}$$

Using the above equations we can try and give two new interpretations to the CLs method.

Given a test of size α, and let $\mu = \mu_{up}$, the power of the test with respect to the BG-only hypothesis is given by $1 - p_b = \phi(\frac{x}{\sigma})$. Keeping the same power, in the CLs method the upper limit is loosen by penalizing and decreasing the size of the test, factorizing it by the power (Eq. 8), i.e. $\alpha' \to \alpha'\phi(\frac{x}{\sigma})$.

The other inteerpretation is keeping the size of the test α and find the constraining power which corresponding to the CL_s limit, $\mu_{up}^{CL_s}$ [2].

Using the PCL formulas we find

$$M_{min} = \phi\left((\frac{x}{\sigma}) + \phi^{-1}(1 - \alpha'\phi(\frac{x}{\sigma})) - \phi^{-1}(1-\alpha)\right) \tag{9}$$

We find $M_{min} = \phi(\frac{x}{\sigma})$ when $\alpha = \alpha'\phi(\frac{x}{\sigma})$. By setting $\alpha = \alpha'$ and keeping the size of the test to α we find that the CLs can be viewed as a PCL with the minimum power dictated by the observed data to be

$$M_{min} = \phi\left((\frac{x}{\sigma}) + \phi^{-1}(1 - \alpha\phi(\frac{x}{\sigma})) - \phi^{-1}(1-\alpha)\right) \tag{10}$$

For $x >> \sigma$ we find $M_{min} = \phi(\frac{x}{\sigma}) = 1 - p_b$ which is the "CL_{s+b}" diagonal line power as expected.

References

[1] Read, Alexander L.,Presentation of search results: The CL(s) technique,J. Phys., G28, 2002, 2693-2704.

[2] G. Cowan, K. Cranmer, E. Gross and O. Vitells, *Power Constrained Limits*, arXiv:1105.3166.