Statistical Data Analysis
Problem Sheet 9
G. Cowan / December 2020

1(a) [3 marks] As the x_i are all independent, the likelihood is found from the product of pdfs,

$$L(v) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi v}} e^{-x_i^2/2v} ,$$

and the log-likelihood is

$$\ln L(v) = -\frac{N}{2} \ln v - \frac{1}{2v} \sum_{i=1}^{N} x_i^2 + C ,$$

where C represents terms that do not depend on v. Setting the derivative of $\ln L$ to zero,

$$\frac{\partial \ln L}{\partial v} = -\frac{N}{2v} + \frac{1}{2v^2} \sum_{i=1}^{N} x_i^2 = 0 ,$$

and solving for v gives the ML estimators,

$$\hat{v} = \frac{1}{N} \sum_{i=1}^{N} x_i^2 .$$

1(b) [2 marks] Using $E[x_i] = 0$ and therefore $E[x_i^2] = v$, the expectation value of $\hat{v}$ is

$$E[\hat{v}] = \frac{1}{N} \sum_{i=1}^{N} E[x_i^2] = v ,$$

and therefore the bias is zero.

1(c) [3 marks] We are told that $t = N\hat{v}/v$ follows a chi-squared distribution for N degrees of freedom, and therefore has variance $2N$. From the variance of t we have

$$V[t] = V\left[\frac{N\hat{v}}{v}\right] = \frac{N^2}{v^2} V[\hat{v}] ,$$

and since $V[t] = 2N$ we find

$$V[\hat{v}] = \frac{2v^2}{N} .$$

1(d) [3] marks] Again using $t = N\hat{v}/v$, we are told to take small values of t as representing lower compatibility. The p-value of v is therefore

$$p_v = P(t \leq t_{\rm obs}|v) = F_{\chi^2_N}(t_{\rm obs})$$

where $F_{\chi^2_N}$ is the chi-squared cumulative distribution for N degrees of freedom, and where $t_{\rm obs}$ refers to the observed value of the statistic. Taking $\hat{v} = v t_{\rm obs}/N$ to mean the observed value of the estimator, we obtain the desired result,

$$p_v = F_{\chi^2_N}(N\hat{v}/v) \ .$$

1(e) [3 marks] To find the upper limit on v, we set the p-value equal to $\alpha = 1 - \mathrm{CL}$ and solve for v, i.e.,

$$p_v = F_{\chi^2_N}(N\hat{v}/v) = \alpha \ .$$

Applying the inverse of the cumulative distribution (the chi-squared quantile) to each side gives

$$\frac{N\hat{v}}{v} = F^{-1}_{\chi^2_N}(\alpha) \ .$$

Solving for v gives the upper limit,

$$v_{\rm up} = \frac{N\hat{v}}{F^{-1}_{\chi^2_N}(\alpha)} \ .$$

1(f) [3 marks] The Jeffreys prior is $\pi_{\rm J}(v) \propto \sqrt{I}$ where I is the Fisher information. For this we need

$$\frac{\partial^2 \ln L}{\partial v^2} = \frac{N}{2v^2} - \frac{1}{v^3}\sum_{i=1}^{N} x_i^2 \ .$$

The Fisher information is therefore

$$I = -E\left[\frac{\partial^2 \ln L}{\partial v^2}\right] = -\frac{N}{2v^2} + \frac{1}{v^3}\sum_{i=1}^{N} E[x_i^2] = \frac{N}{2v^2} \ ,$$

where to obtain the final equality we used $E[x_i^2] = v$. Therefore the Jeffreys prior is

$$\pi_{\rm J}(v) \propto \sqrt{I} \propto \frac{1}{v} \ .$$

1(g) [3 marks] Using Bayes' theorem as a proportionality, the posterior probability for v given $\vec{x} = (x_1, \ldots, x_N)$ is

$$p(v|\vec{x}) \propto p(\vec{x}|v)\pi_{\rm J}(v) \propto \frac{1}{v}\prod_{i=1}^{N} \frac{1}{\sqrt{2\pi v}} e^{-x_i^2/2v} \propto v^{-N/2-1} \exp\left[-\frac{1}{2v}\sum_{i=1}^{N} x_i^2\right] \ .$$

To find the posterior mode we set its derivative to zero. Letting $Q = \sum_{i=1}^{N} x_i^2$ we obtain

$$\frac{\partial p(v|\vec{x})}{\partial v} \propto v^{-N/2-1} e^{Q} \frac{Q}{2v^2} + e^{Q} \left(-\frac{N}{2} - 1 \right) v^{-N/2-2} = 0 \, .$$

Solving for v gives the mode,

$$\text{mode}[v] = \frac{1}{N+2} \sum_{i=1}^{N} x_i^2 \, .$$